Notes on LALDemod

Xavier Siemens
(Dated: September 16, 2004)

I. THE DISCRETISED $\mathcal{F}$-STATISTIC

For a set of template parameters $\lambda = (f_0, f_1, ..., \alpha, \delta, ...)$ the $\mathcal{F}$-statistic defined in Eq. (110) of Jaranowski, Krolak, and Schutz (gr-qc/9804014) is given by

$$\mathcal{F}(\lambda) = \frac{4}{S_h(f_0)T_0} \frac{B|F_a(\lambda)|^2 + A|F_b(\lambda)|^2 - 2C\Re(F_a(\lambda)F_b^*(\lambda))}{D}, \tag{1}$$

where T_0 is the observation time, $S_h(f_0)$ is one-sided spectral density of the noise,

$$F_a(\lambda) = \int_{-T_o/2}^{T_o/2} x(t)a(t)\exp[-2\pi i\Phi(t,\lambda)]\,dt, \tag{2}$$

$$F_b(\lambda) = \int_{-T_o/2}^{T_o/2} x(t)b(t)\exp[-2\pi i\Phi(t,\lambda)]\,dt, \tag{3}$$

$A := (a||a)$, $B := (b||b)$, $C := (a||b)$ and $D = AB - C^2$ with

$$(x||y) := \frac{2}{T_o}\int_{-T_o/2}^{T_o/2} x(t)y(t)\,dt. \tag{4}$$

LALDemod computes a discretised version of the above expressions for F_a and F_b,

$$F_a = \sum_{i=0}^{NM-1} x_i a_i e^{-2\pi i\Phi_i}\Delta t, \tag{5}$$

$$F_b = \sum_{i=0}^{NM-1} x_i b_i e^{-2\pi i\Phi_i}\Delta t. \tag{6}$$

We divide the data into M short chunks of length T_{SFT} each with N points. Note that $T_0 = MT_{SFT}$. The length of these chunks is such that the amplitude modulation functions $a(t)$ and $b(t)$ do not change significantly so that we can write,

$$F_a = \sum_{\alpha=0}^{M-1} a_\alpha \sum_{j=0}^{N-1} x_{\alpha,j} e^{-2\pi i\Phi_{\alpha,j}}\Delta t, \tag{7}$$

$$F_b = \sum_{\alpha=0}^{M-1} b_\alpha \sum_{j=0}^{N-1} x_{\alpha,j} e^{-2\pi i\Phi_{\alpha,j}}\Delta t. \tag{8}$$

The index in the sum above is $i = N\alpha + j$. Note that we can also write,

$$A = \frac{2}{M}\sum_{\alpha=0}^{M-1} a_\alpha a_\alpha, \tag{9}$$

$$B = \frac{2}{M}\sum_{\alpha=0}^{M-1} b_\alpha b_\alpha, \tag{10}$$

$$C = \frac{2}{M}\sum_{\alpha=0}^{M-1} a_\alpha b_\alpha. \tag{11}$$

$$\tag{12}$$

Our data set is given in terms of Fourier transforms of these short data chunks (SFTs),

$$X_{\alpha,k} = \sum_{k=0}^{N-1} x_{\alpha,j}\, e^{-2\pi ijk/N}. \tag{13}$$

The inverses of the Fourier transforms are

$$x_{\alpha,j} = \frac{1}{N}\sum_{k=0}^{N-1} X_{\alpha,k}\, e^{2\pi ijk/N}. \tag{14}$$

If we substitute this expression into, say F_a in Eq. 7, we obtain,

$$F_a = \frac{\Delta t}{N}\sum_{\alpha=0}^{M-1} a_\alpha \sum_{k=0}^{N-1} X_{\alpha,k} \sum_{j=0}^{N-1} e^{-2\pi i(\Phi_{\alpha,j}-jk/N)}. \tag{15}$$

There is a similar expression for F_b. In the following we will proceed with the calculations using F_a; the corresponding results for F_b are completely analogous.

We assume the phase evolution is linear within a short chunk and Taylor expand the phase model about the middle of each of the chunk,

$$\Phi_{\alpha,j} = \Phi_{\alpha,1/2} + \dot{\Phi}_{\alpha,1/2}(t_{\alpha,j} - t_{\alpha,1/2}). \tag{16}$$

Here, $t_{\alpha,j} = (\alpha N + j)\Delta t$, so that,

$$\Phi_{\alpha,j} = \Phi_{\alpha,1/2} + \dot{\Phi}_{\alpha,1/2}(j - N/2)\Delta t = \Phi_{\alpha,1/2} + \dot{\Phi}_{\alpha,1/2}(j/N - 1/2)T_{SFT}. \tag{17}$$

The Taylor expansion of the phase model allows us to write Eq. 15 as,

$$\begin{aligned} F_a &= \frac{\Delta t}{N}\sum_{\alpha=0}^{M-1} a_\alpha \sum_{k=0}^{N-1} X_{\alpha,k} \sum_{j=0}^{N-1} e^{-2\pi i(\Phi_{\alpha,1/2}+\dot{\Phi}_{\alpha,1/2}(j/N-1/2)T_{SFT}-jk/N)}. \\ &= \frac{\Delta t}{N}\sum_{\alpha=0}^{M-1} a_\alpha e^{-2\pi i(\Phi_{\alpha,1/2}-\dot{\Phi}_{\alpha,1/2}T_{SFT}/2)} \sum_{k=0}^{N-1} X_{\alpha,k} \sum_{j=0}^{N-1} e^{-2\pi i(\dot{\Phi}_{\alpha,1/2}T_{SFT}-k)j/N}. \end{aligned} \tag{18}$$

The last sum in this expression can be evaluated analytically. In particular,

$$\sum_{j=0}^{N-1} z^{cj} = \frac{1-z^{Nc}}{1-z^{c}}. \tag{19}$$

We take $z = e$, $c = -ix/N$, $x = 2\pi(\dot{\Phi}_{\alpha,1/2}T_{SFT} - k)$, so that the sum is given by,

$$\sum_{j=0}^{N-1} e^{-ixj/N} = \frac{1-e^{-ix}}{1-e^{-ix/N}} \tag{20}$$

In the large N limit the exponent of the denominator will be small so that

$$\frac{1-e^{-ix}}{1-e^{-ix/N}} \approx \frac{1-e^{-ix}}{1-(1-ix/N)} = \frac{iN}{x}(e^{-ix}-1) = N(\frac{\sin x}{x} - i\frac{1-\cos x}{x}) \tag{21}$$

So we can write,

$$F_a \approx \Delta t \sum_{\alpha=0}^{M-1} a_\alpha e^{-2\pi iy} \sum_{k=0}^{N-1} X_{\alpha,k}P_{\alpha,k} \tag{22}$$

with

$$y = \Phi_{\alpha,1/2} - \dot{\Phi}_{\alpha,1/2}T_{SFT}/2 \tag{23}$$

and the Dirichlet kernel,

$$P_{\alpha,k} = \sin x/x - i(1-\cos x)/x, \tag{24}$$

and

$$x = 2\pi(\dot{\Phi}_{\alpha,1/2}T_{SFT} - k). \tag{25}$$

The function $P_{\alpha,k}$ is very strongly peaked around $x = 0$, which is near a value of the frequency index $k^* = \text{floor}(\dot{\Phi}_{\alpha,1/2}T_{SFT})$. This means one only needs to evaluate the sum over k for a few terms Δk around k^*. With this in mind we write

$$F_a \approx \Delta t \sum_{\alpha=0}^{M-1} a_\alpha e^{-2\pi i y} \sum_{k=k^*-\Delta k}^{k^*+\Delta k} X_{\alpha,k}P_{\alpha,k}. \tag{26}$$

Similarly

$$F_b \approx \Delta t \sum_{\alpha=0}^{M-1} b_\alpha e^{-2\pi i y} \sum_{k=k^*-\Delta k}^{k^*+\Delta k} X_{\alpha,k}P_{\alpha,k}. \tag{27}$$

II. NORMALISATION OF THE DATA

Because the noise may vary from SFT to SFT, due to non-stationarity, and between frequency bins when the noise is coloured we normalise our SFT data to absorb the $1/Sh(f_0)$ term in the definition of the $\mathcal{F}$-statistic.

In particular we take,

$$X_{\alpha,k} \longrightarrow X'_{\alpha,k} = \frac{X_{\alpha,k}}{\sqrt{S_{\alpha,k}T_{SFT}}} \tag{28}$$

where $S_{\alpha,k}$ is an estimate of the one-sided power spectral density for the kth frequency bin of the αth SFT.

This means that in terms of the dimensionless

$$\tilde{X}_{\alpha,k} = \Delta t X'_{\alpha,k} \tag{29}$$

we can write

$$\tilde{F}_a \approx \sum_{\alpha=0}^{M-1} a_\alpha e^{-2\pi i y} \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{X}_{\alpha,k}P_{\alpha,k}, \tag{30}$$

and

$$\tilde{F}_b \approx \sum_{\alpha=0}^{M-1} b_\alpha e^{-2\pi i y} \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{X}_{\alpha,k}P_{\alpha,k}, \tag{31}$$

and thus

$$\mathcal{F} = \frac{4}{M}\frac{B|\tilde{F}_a|^2 + A|\tilde{F}_b|^2 - 2C\Re(\tilde{F}_a\tilde{F}_b^*)}{D}. \tag{32}$$

This is the quantity that is actually computed by LALDemod.

III. THE PHASES $\Phi_{\alpha,1/2}$ AND $\dot{\Phi}_{\alpha,1/2}$

Now, if one adopts the notation $\Delta T_\alpha \equiv \left[T(t_{\alpha,1/2}) - T(t_0)\right]$ and $\dot{T}_\alpha \equiv dT/dt(t_{\alpha,1/2})$ the phase terms in the above equation are given by (this is from the ComputeSky.c documentation)

$$\Phi_{\alpha,1/2} = f_0\Delta T_\alpha + \frac{1}{2}f_1\Delta T_\alpha^2 + \frac{1}{3}f_2\Delta T_\alpha^3 + \frac{1}{4}f_3\Delta T_\alpha^4 + \frac{1}{5}f_4\Delta T_\alpha^5 + \frac{1}{6}f_5\Delta T_\alpha^6 + ... \tag{33}$$

$$\dot{\Phi}_{\alpha,1/2} = \dot{T}_\alpha\left(f_0 + f_1\Delta T_\alpha + f_2\Delta T_\alpha^2 + f_3\Delta T_\alpha^3 + f_4\Delta T_\alpha^4 + f_5\Delta T_\alpha^5 + ...\right). \tag{34}$$

These constants, for each value of α, require $\dot{T}_\alpha$ and ΔT_α, which are calculated by a suitable timing routine.

In terms of these, x is given by,

$$\begin{aligned} x &= 2\pi(f_0\dot{T}_\alpha T_{SFT} + f_1\dot{T}_\alpha T_{SFT}\Delta T_\alpha + f_2\dot{T}_\alpha T_{SFT}\Delta T_\alpha^2 + ... - k) \\ &= 2\pi(\sum_{s=0}^{n_{spin}} f_s T_{SFT}\dot{T}_\alpha \Delta T_\alpha^s - k) = 2\pi(\sum_{s=0}^{n_{spin}} f_s B_{s\alpha} - k). \end{aligned} \tag{35}$$

The $B_{s\alpha}$ are called sky-constants in various codes.

In terms of these, y is given by,

$$\begin{aligned} y &= f_0(\Delta T_\alpha - \frac{1}{2}\dot{T}_\alpha T_{SFT}) + f_1(\frac{1}{2}\Delta T_\alpha^2 - \frac{1}{2}\frac{1}{2}\dot{T}_\alpha T_{SFT}\Delta T_\alpha) + ... \\ &= \sum_{s=0}^{n_{spin}} f_s(\frac{1}{s+1}\Delta T_\alpha^{s+1} - \frac{1}{2}T_{SFT}\dot{T}_\alpha \Delta T_\alpha^s) = \sum_{s=0}^{n_{spin}} f_s A_{s\alpha}. \end{aligned} \tag{36}$$

The $A_{s\alpha}$, like the $B_{s\alpha}$ are also called sky-constants.

IV. LALDEMOD FUNCTION DETAILS

LALDemod computes the $\mathcal{F}$-statistic for a frequency band given some template parameters.

The first thing LALDemod does is to set up a look-up table that is used in the calculation of the Dirichlet kernel, Eq. 21.

Then two arrays called xSum and ySum are computed. These arrays contain the phase evolution of the signal that depends only on the spindown values (not the frequency). In particular,

$$\text{xSum[alpha]} = \sum_{s=1}^{n_{spin}} f_s B_{s\alpha} \tag{37}$$

and

$$\text{ySum[alpha]} = \sum_{s=1}^{n_{spin}} f_s A_{s\alpha}. \tag{38}$$

Note that both sums start at 1.

The first loop is a loop over frequencies, for each of these frequencies F_a and F_b will be evaluated and from them the $\mathcal{F}$-statistic computed.

Once the frequency is set, there is a loop in α over the SFTs. This loop corresponds to the outermost sum (the one over α) in the expressions for F_a and F_b. For each α a pointer to the SFT data (variable Xalpha in the code), and values for the amplitude modulation parameters a_α (called a in the code) and b_α (called b in the code) are assigned.

The line

$$\text{xTemp} = \text{f} * \text{skyConst[tempInt1[alpha]]} + \text{xSum[alpha]}; \tag{39}$$

computes the sum $\sum_{s=0}^{n_{spin}} f_s B_{s\alpha}$. The term skyConst[tempInt1[alpha]] $= \dot{T}_\alpha T_{SFT}$ and xSum[alpha] $= \sum_{s=1}^{n_{spin}} f_s B_{s\alpha}$. This variable is a real number however it is dimensionless and "index-like" (like $f * T_{SFT}$).

Next the values of the sines and cosines in the Dirichlet kernel are computed. They need not be computed for every k in the inner-most loop because the variable x in that loop (or in Eq. 26) will only vary by integer amounts (and the sines and cosines will not change). Hence, they are only calculated once for a single value: the difference between the value of xTemp and its integer part. Notice that what is called tcos in the code is actually $\cos x - 1$.

Finally, before the inner-most loop, the variable tempFreq is re-assigned to be the angular distance between xTemp and the smallest value of the frequency index that will contribute to the sum over terms in the Dirichlet kernel (the number of these terms on either side of xTemp is params->Dterms). Also, the variable k1 is set to the frequency index that corresponds to that distance.

The loop over terms in the Dirichlet kernel (loop over k), starts from the smallest value of the frequency index that contributes to the calculation and increases to the largest value of the frequency index that contributes to the calculation. Both of these bounds depend only on the number of terms we have decided to keep in the Dirichlet

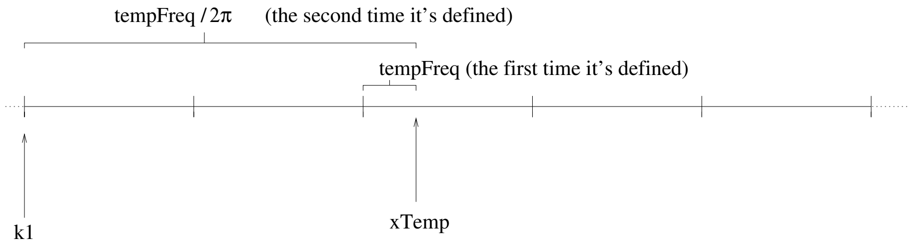

FIG. 1: Illustration of variable setup before the innermost loop of LALDemod when params->Dterms=3. Frequency bins of the SFT are shown as the small vertical lines with frequency increasing from left to right.

kernel. In this loop sftIndex is the index that corresponds to the actual data that was passed to LALDemod (since the entire band of the SFT was not passed); params->ifmin contains the smallest value of the frequency index of the original SFT data that is passed to LALDemod. The loop then combines the frequency bins that contribute to the signal according to Eq. 26.

For example Figure 1 illustrates how this inner-most loop works with params->Dterms=3. The first time the variable tempFreq is defined it is given by the difference between xTemp and its integer part. Then tempFreq is defined to be the angular distance between xTemp and the smallest value of the frequency that will contribute to the sum over k. Finally k1 is set to the smallest value of the frequency index that contributes to the sum, in this case the integer part of xTemp-2. Then the look over k begins. For k=0 the value of x is the value of tempFreq, and we divide the previously computed $\sin x$ and $\cos x - 1$ by x to compute the Dirichlet kernel. Then the value of the index of the data that is actually passed to LALDemod is computed, the data is assigned to that point and the product of $X_{\alpha,k} P_{\alpha,k}$ is computed. For k=1, the value of tempFreq is decreased by 2π and so on...

After this loop the line

$$\mathrm{y} = -\mathrm{LALTWOPI} * (\mathrm{f} * \mathrm{skyConst[tempInt1[alpha]} - 1] + \mathrm{ySum[alpha]}); \tag{40}$$

computes y (this time with an extra factor of 2π). The term $\mathrm{skyConst[tempInt1[alpha]} - 1] = \Delta T_\alpha - \frac{1}{2}\dot{T}_\alpha T_{SFT}$ and $\mathrm{ySum[alpha]} = \sum_{s=1}^{n_{spin}} f_s A_{s\alpha}$.

And finally the amplitude modulation is folded in to evaluate F_a and F_b. When the loop over α (the SFTs) is finished the $\mathcal{F}$-statistic computed and we proceed to the next value of the frequency.