Note on the SEOBNRv5HM waveform model

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This note is based on one of the papers we are writing on the SEOBNRv5 waveform model. It summarizes the structure of the aligned-spin Hamiltonian, and explains the construction of the multipolar waveform modes of the SEOBNRv5HM model, including the calibration to numerical relativity. Please keep in mind, when reading this note, that citations are not complete. The note has been written with the only scope of facilitating the review of the SEOBNRv5 model for O4.

NOTATION

We consider a binary with masses m_1 and m_2, with $m_1 \geq m_2$, and spins $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. We define the following combinations of the masses:

$$\begin{aligned} M &\equiv m_1 + m_2, \qquad \mu \equiv \frac{m_1 m_2}{M}, \qquad \nu \equiv \frac{\mu}{M}, \\ \delta &\equiv \frac{m_1 - m_2}{M}, \qquad q \equiv \frac{m_1}{m_2}, \end{aligned} \tag{1}$$

and define the dimensionless spin vectors

$$\boldsymbol{\chi}_{\rm i} \equiv \frac{\boldsymbol{a}_{\rm i}}{m_{\rm i}} = \frac{\boldsymbol{S}_{\rm i}}{m_{\rm i}^2}, \tag{2}$$

along with the intermediate definition for $\boldsymbol{a}_{\rm i}$, where i = 1,2. The spin magnitudes $\chi_{\rm i}$ vary between -1 and 1, with positive spins being in the direction of the angular momentum. We also define the following combinations of spins:

$$\begin{aligned} \boldsymbol{S} &\equiv \boldsymbol{S}_1 + \boldsymbol{S}_2, \qquad \boldsymbol{S}_* \equiv \frac{m_2}{m_1}\boldsymbol{S}_1 + \frac{m_1}{m_2}\boldsymbol{S}_2, \\ \chi_S &\equiv \frac{\chi_1 + \chi_2}{2}, \qquad \chi_A \equiv \frac{\chi_1 - \chi_2}{2}, \\ \boldsymbol{a}_\pm &\equiv \frac{\boldsymbol{a}_1 \pm \boldsymbol{a}_2}{M} = \frac{m_1}{M}\chi_1 \pm \frac{m_2}{M}\chi_2. \end{aligned} \tag{3}$$

Note that, unlike $\boldsymbol{a}_{\rm i}$, we define $\boldsymbol{a}_\pm$ to be dimensionless by dividing $\boldsymbol{a}_{\rm i}$ by the total mass.

The relative position and momentum vectors are denoted $\boldsymbol{R}$ and $\boldsymbol{P}$, with

$$\boldsymbol{P}^2 = P_R^2 + \frac{L^2}{R^2}, \qquad P_R = \boldsymbol{n}\cdot\boldsymbol{P}, \qquad \boldsymbol{L} = \boldsymbol{R}\times\boldsymbol{P}, \tag{4}$$

where $\boldsymbol{n} = \boldsymbol{R}/R$, and $\boldsymbol{L}$ is the orbital angular momentum with magnitude L. The total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$. For precessing spins, we use the spherical-coordinates phase-space variables $(R, \theta, \phi, P_R, P_\theta, P_\phi)$, where θ is the polar angle, ϕ is the azimuthal angle, and P_ϕ and P_θ are their conjugate momenta. For equatorial orbits (aligned-spins), the angular momentum $L = P_\phi$.

We use the rescaled dimensionless variables

$$t \equiv \frac{T}{M}, \qquad \boldsymbol{r} \equiv \frac{\boldsymbol{R}}{M}, \qquad u \equiv \frac{1}{r}, \qquad \bar{\boldsymbol{L}} = \frac{\boldsymbol{L}}{M\mu}, \qquad \boldsymbol{p} \equiv \frac{\boldsymbol{P}}{\mu}, \tag{5}$$

$$p_r \equiv \frac{P_R}{\mu}, \qquad p_\theta \equiv \frac{P_\theta}{M\mu}, \qquad p_\phi \equiv \frac{P_\phi}{M\mu}, \qquad \bar{H} \equiv \frac{H}{\mu}, \tag{6}$$

where we use a lowercase symbol to indicate the dimensionless quantities, except for the dimensionless angular momentum $\bar{\boldsymbol{L}}$ and Hamiltonian $\bar{H}$.

We use units in which $c = G = 1$.

I. THE SEOBNRV5 HAMILTONIAN

In the EOB formalism [1–5] the dynamics of the BH binary is mapped to that of the effective problem of a test particle in a deformed Schwarzschild or Kerr background, with the deformation being parametrized by the symmetric mass ratio ν. The energy map relating the effective Hamiltonian $H_{\rm eff}$ and the two-body EOB Hamiltonian $H_{\rm EOB}$ is given by

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}. \tag{7}$$

The generic-spin SEOBNRv5 Hamiltonian is based on that of a test mass in Kerr [6, 7]. By contrast the generic-spin SEOBNRv4 [8–10] Hamiltonian was based on the one of a spinning test-body in a Kerr background [11–13].

The generic-spin SEOBNRv5 Hamiltonian includes most of the 5PN nonspinning contributions [14], together with spin-orbit (SO) information up to the next-to-next-to-leading order (NNLO), spin-spin (SS) information to NNLO, as well as cubic- and quartic-in-spin terms at leading order (LO), corresponding to all PN information up to 4PN order for precessing spins. More details about the derivation of the generic-spin Hamiltonian, together with the full expressions, are given in Ref. [7]. Here we summarize the structure of the aligned-spin Hamiltonian, highlighting where NR calibration parameters enter the expressions.

A. Zero-spin Hamiltonian

The effective Hamiltonian in the zero-spin (point-mass) limit can be written as

$$\bar{H}_{\rm eff,pm} = \sqrt{A_{\rm pm}\left[1 + A_{\rm pm}\bar{D}_{\rm pm}p_r^2 + p_\phi^2 u^2 + Q_{\rm pm}\right]}. \tag{8}$$

For the potentials A, $\bar{D}$, and Q, we use the 5PN results of Ref. [14], which are missing two quadratic-in-ν coefficients in

A and $\bar{D}$. The 5PN Taylor-expanded potential A is given by

$$
\begin{aligned}
A^{\rm Tay}_{\rm pm}(u) = {} & 1-2u+2\nu u^3+\nu\left(\frac{94}{3}-\frac{41\pi^2}{32}\right)u^4 \\
& +\left[\nu\left(\frac{2275\pi^2}{512}-\frac{4237}{60}+\frac{128\gamma_E}{5}+\frac{256\ln 2}{5}\right)\right. \\
& \left.+\left(\frac{41\pi^2}{32}-\frac{221}{6}\right)\nu^2+\frac{64}{5}\nu\ln u\right]u^5 \\
& +\left[\nu a_6+\left(-\frac{144\nu^2}{5}-\frac{7004\nu}{105}\right)\ln u\right]u^6, \qquad (9)
\end{aligned}
$$

where we replaced the coefficient of u^6 in $A(r)$, except for the log part, by the parameter a_6, which is calibrated to quasi-circular NR simulations. Note that we pull out a factor of ν from a_6 compared to the definition in Ref. [14]. Then, we perform a (1,5) Padé resummation of $A^{\rm Tay}_{\rm pm}(u)$, while treating $\ln u$ as a constant, i.e., we use

$$
A_{\rm pm} = P^1_5[A^{\rm Tay}_{\rm pm}(u)]. \qquad (10)
$$

For the $\bar{D}_{\rm pm}$ potential we use again 5PN results, and set the remaining unknown coefficient $d_5^{\nu^2}$ to zero. We then perform a (2,3) Padé resummation of $\bar{D}^{\rm Tay}_{\rm pm}(u)$

$$
\bar{D}_{\rm pm} = P^2_3[\bar{D}^{\rm Tay}_{\rm pm}(u)]. \qquad (11)
$$

The 5.5PN contributions to A and $\bar{D}$ are known [14]; however, since we Padé resum these potentials, we find it more convenient to stop at 5PN.

For the Q potential, we use the full 5.5PN expansion, which is expanded in eccentricity to $O(p_r^8)$.

The calibration parameter a_6 is a function of ν; to determine its value in the limit $\nu\to 0$, we use the gravitational-self-force results of Refs. [15, 16] for the frequency shift of the innermost stable circular orbit (ISCO), which is

$$
\begin{aligned}
M\Omega^{\rm 1SF}_{\rm ISCO} &= 6^{-3/2}(1+C_\Omega/q), \\
C_\Omega &= 1.25101539\pm 4\times 10^{-8}.
\end{aligned} \qquad (12)
$$

The ISCO can be computed from the Hamiltonian by solving $\partial H/\partial r = 0 = \partial^2 H/\partial r^2$ for r and p_ϕ with $p_r = 0$. The value of a_6 that gives best agreement with $\Omega^{\rm 1SF}_{\rm ISCO}$ is

$$
a_6|_{\nu\to 0} \simeq 39.1. \qquad (13)
$$

B. Aligned spins Hamiltonian

For aligned spins, the effective Hamiltonian reduces to the equatorial Kerr Hamiltonian [6, 17], and to include higher PN information, we use the following ansatz:

$$
\begin{aligned}
\bar{H}_{\rm eff} = {} & \frac{1}{r^3+a_+^2(r+2)}\left[p_\phi(g_{a_+}a_+ + g_{a_-}\delta a_-)+{\rm SO}_{\rm calib}+G_{a^3}\right] \\
& +\left[A\left(1+p^2+B_{np}p_r^2+B^{\rm Kerr,eq}_{npa}\frac{p_\phi^2 a_+^2}{r^2}+Q\right)\right]^{1/2}, \qquad (14)
\end{aligned}
$$

where g_{a_+} and g_{a_-} include the SO corrections up to 3.5PN, ${\rm SO}_{\rm calib}$ is an NR calibration term at 4.5PN of the form

$$
{\rm SO}_{\rm calib} = d_{\rm SO}\frac{\nu}{r^3}p_\phi a_+, \qquad (15)
$$

and G_{a^3} contains S^3 corrections [7]. The nonspinning and SS contributions are included in A, B_{np} and Q, while the S^4 corrections are added in A. The potential $B^{\rm Kerr,eq}_{npa}$ is kept the same as in the Kerr Hamiltonian.

The gyro-gravitomagnetic factors g_{a_+} and g_{a_-} in the SO part of the Hamiltonian are often chosen to be in a gauge such that they are functions of $1/r$ and p_r^2 only [18, 19]. However, in `SEOBNRv5`, we find better results when using a gauge in which g_{a_+} and g_{a_-} depend on $1/r$ and $\bar{L}^2/r^2$, but not on p_r^2. The 4.5PN SO coupling was derived in Refs. [20–23], and can be included in the effective Hamiltonian. However, we found that using a calibration term at 5.5PN had a small effect on the dynamics, and thus only included the 3.5PN information with a 4.5PN calibration term.

C. Hamiltonians in tortoise coordinates

The tortoise-coordinate r_* is defined by [24, 25]

$$
\frac{dr_*}{dr} = \frac{1}{\xi(r)}, \qquad \xi(r)\equiv A_{\rm pm}(r)\sqrt{\bar{D}_{\rm pm}(r)}. \qquad (16)
$$

The conjugate momentum to r_* is p_{r_*}, which is given by the relation

$$
p_{r_*} = p_r\xi(r). \qquad (17)
$$

The nonspinning effective Hamiltonian (8) in terms of p_{r_*} takes the simpler form

$$
\bar{H}_{\rm eff,pm} = \sqrt{p_{r_*}^2 + A(r)\left[1+p_\phi^2u^2+Q(r,p_{r_*})\right]}, \qquad (18)
$$

where we obtain $Q(r,p_{r_*})$ by converting p_r to p_{r_*} using Eq. (17), then PN expand to 5.5PN order.

For both aligned and precessing spins, a convenient choice for ξ is

$$
\xi(r) = \frac{\bar{D}^{1/2}_{\rm pm}\left(A_{\rm pm}+a_+^2u^2\right)}{1+a_+^2u^2}, \qquad (19)
$$

which is similar to what was used for ξ in `SEOBNRv4` [25, 26] except for the different resummation and PN orders in $A_{\rm pm}$ and $\bar{D}_{\rm pm}$. In the $\nu\to 0$ limit, ξ reduces to the Kerr value $(dr/dr_*)_{\rm Kerr} = (r^2-2r+a_+^2)/(r^2+a_+^2)$.

II. THE SEOBNRV5 MULTIPOLAR WAVEFORMS

The complex linear combination of GW polarizations, $h(t)\equiv h_+(t)-ih_\times(t)$ is expanded in the basis of -2 spin-weighted spherical harmonics [27] as follows:

$$
h(t;\boldsymbol{\lambda},\iota,\varphi_c) = \sum_{\ell\geq 2}\sum_{|m|\leq\ell} {}_{-2}Y_{\ell m}(\iota,\varphi_c)\,h_{\ell m}(t;\boldsymbol{\lambda}), \qquad (20)
$$

where $\boldsymbol{\lambda}$ denotes the intrinsic parameters of the compact binary source, such as masses ($m_{1,2}$) and spins ($\boldsymbol{\chi}_{1,2}$). In the aligned-spin case there are only three parameters (q,χ_1,χ_2), since the waveforms scale trivially with the total mass M. The parameters (ι,φ_c) specify the direction of the incoming GW radiation and are called inclination and coalescence phase, respectively.

In this Section we describe the building blocks used in the construction of the multipolar spinning, nonprecessing waveform modes $h_{\ell m}$. We closely follow [9], and highlight differences compared to `SEOBNRv4HM` when needed. In the EOB framework the gravitational wave modes defined in Eq. (20) are decomposed in inspiral-plunge modes and merger-ringdown (MR) modes. In `SEOBNRv5HM` we model the (2,2) and the largest subdominant modes [9] (3,3), (2,1), (4,4), (3,2), (5,5) and (4,3). The generic mode is written as:

$$h_{\ell m}(t) = \begin{cases} h_{\ell m}^{\text{insp-plunge}}(t), & t < t_{\text{match}}^{\ell m} \\ h_{\ell m}^{\text{merger-RD}}(t), & t > t_{\text{match}}^{\ell m} \end{cases} \tag{21}$$

where we define $t_{\text{match}}^{\ell m}$ as

$$t_{\text{match}}^{\ell m} = \begin{cases} t_{\text{peak}}^{22}, & (\ell,m) = (2,2),(3,3), \\ & \quad (2,1),(4,4), \\ & \quad (3,2),(4,3) \\ t_{\text{peak}}^{22} - 10M, & (\ell,m) = (5,5), \end{cases} \tag{22}$$

with t_{peak}^{22} being the peak of the (2,2) mode amplitude. The choice of a different attachment point for the (5,5) mode is motivated, as in [9], by the fact that $t_{\text{peak}}^{55} - t_{\text{peak}}^{22} > 0$, and at late times the error in some of the NR waveforms is too large to accurately extract the quantities of interest.

A. Inspiral-plunge $h_{\ell m}$ modes and radiation-reaction force

The inspiral-plunge EOB waveform modes can be written as

$$h_{\ell m}^{\text{insp-plunge}} = h_{\ell m}^{\text{F}} N_{\ell m} \tag{23}$$

where $h_{\ell m}^{\text{F}}$ is a factorized, resummed form of the PN GW modes for aligned-spins in circular orbits [24, 28–30], while $N_{\ell m}$ is the nonquasi-circular (NQC) correction, aimed at incorporating radial effects that are relevant at the end of the inspiral. The factorized modes are written as

$$h_{\ell m}^{\text{F}} = h_{\ell m}^{(N,\epsilon_{\ell m})} \hat{S}_{\text{eff}}^{(\epsilon_{\ell m})} T_{\ell m} f_{\ell m} e^{i\delta_{\ell m}}. \tag{24}$$

The first factor, $h_{\ell m}^{(N,\epsilon_p)}$ is the leading (Newtonian) order waveform and its explicit expression is [29]

$$h_{\ell m}^{\text{N}} = \frac{\nu M}{D_L} n_{\ell m} c_{\ell+\epsilon_{\ell m}}(\nu) v_\phi^{\ell+\epsilon_{\ell m}} Y_{\ell-\epsilon_{\ell m},-m}\left(\frac{\pi}{2},\phi\right). \tag{25}$$

Here D_L is the luminosity distance, $Y_{\ell m}$ is a scalar spherical harmonic, $\epsilon_{\ell m}$ is the parity of the mode,

$$\epsilon_{\ell m} = \begin{cases} 0, & \ell+m \text{ is even} \\ 1, & \ell+m \text{ is odd}, \end{cases} \tag{26}$$

and the functions $n_{\ell m}$ and $c_k(\nu)$ are given by

$$n_{\ell m} = \begin{cases} \dfrac{8\pi(im)^\ell}{(2\ell+1)!!}\sqrt{\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)}}, & \ell+m \text{ is even} \\ \dfrac{-16i\pi(im)^\ell}{(2\ell+1)!!}\sqrt{\frac{(2\ell+1)(\ell+2)(\ell^2-m^2)}{(2\ell-1)(\ell+1)\ell(\ell-1)}}, & \ell+m \text{ is odd}, \end{cases} \tag{27}$$

and

$$c_k(\nu) = \left(\frac{1-\sqrt{1-4\nu}}{2}\right)^{k-1} + (-1)^k\left(\frac{1+\sqrt{1-4\nu}}{2}\right)^{k-1}. \tag{28}$$

Finally, v_ϕ in (25) is given by

$$v_\phi = M\Omega r_\Omega, \tag{29}$$

where Ω is the orbital frequency and

$$r_\Omega = \left(\frac{\partial H_{\text{EOB}}}{\partial p_\phi}\right)^{-2/3}\Bigg|_{p_r=0}. \tag{30}$$

The (dimensionless) effective source term $\hat{S}_{\text{eff}}$ is given by either the effective energy E_{eff} or the orbital angular momentum p_ϕ, both expressed as functions of $v_\Omega \equiv (M\Omega)^{1/3} = \sqrt{x}$, such that

$$\hat{S}_{\text{eff}} = \begin{cases} \frac{E_{\text{eff}}(v_\Omega)}{\mu}, & \ell+m \text{ even} \\ v_\Omega \frac{p_\phi(v_\Omega)}{\mu M}, & \ell+m \text{ odd} \end{cases}, \tag{31}$$

where E_{eff} is related to the total energy E via the EOB energy map $E = M\sqrt{1+2\nu(E_{\text{eff}}/\mu-1)}$. The factor $T_{\ell m}$ resums the infinite number of "leading logarithms" entering the tail effects [31], and is given by

$$T_{\ell m} = \frac{\Gamma\left(\ell+1-2i\hat{k}\right)}{\Gamma(\ell+1)} e^{\pi\hat{k}} e^{2i\hat{k}\ln(2m\Omega r_0)}, \tag{32}$$

where $\Gamma(...)$ is the Euler gamma function, $\hat{k} \equiv m\Omega E$ and the constant r_0 takes the value $2M/\sqrt{e}$ to give agreement with waveforms computed in the test-body limit [29].

The remaining part of the factorized modes is expressed as an amplitude $f_{\ell m}$ and a phase $\delta_{\ell m}$, which are computed such that the expansion of $h_{\ell m}^{\text{F}}$ agrees with the PN-expanded modes. To improve the agreement with numerical-relativity waveforms, $f_{\ell m}$ is further resummed as [29, 30] $f_{\ell m} = (\rho_{\ell m})^\ell$ to reduce the magnitude of the 1PN non-spinning coefficient, which grows linearly with ℓ. For spinning binaries, the non-spinning and spin contributions are separated for the odd m modes, such that

$$f_{\ell m} = \begin{cases} \rho_{\ell m}^\ell, & m \text{ even} \\ (\rho_{\ell m}^{\text{NS}})^\ell + f_{\ell m}^{\text{S}}, & m \text{ odd} \end{cases}, \tag{33}$$

where $\rho_{\ell m}^{\text{NS}}$ is the non-spinning part of $\rho_{\ell m}$, while $f_{\ell m}^{\text{S}}$ is the spin part of $f_{\ell m}$.

As in `SEOBNRv4HM`, the presence of minima in the amplitude for some of the modes needs to be treated, before applying the NQC corrections, by additional calibration parameters. The modes for which this is needed are the (2,1), (5,5) and $(4,3)$. The minima occur for $q \sim 1$ and large $|\chi_A|$, and can lead to unphysical features in the amplitude after applying the NQC corrections if they occur close to the attachment point $t \sim t_{\rm match}$. For the $(2,1)$ mode this behavior is also found in NR simulations, while for the $(5,5)$ and $(4,3)$ this is not observed in the NR simulations at our disposal, and is likely an artifact of the PN modes [9].

The calibration term takes the form $c_{\ell m} v_\Omega^{\beta_{\ell m}}$, where $\beta_{\ell m}$ denotes the first-order term at which the PN series of $h^{\rm PN}_{\ell m}$ is not known today with its complete dependence on mass ratio and spins, and is included in $f_{\ell m}$. The calibration parameter $c_{\ell m}$ is evaluated to satisfy the condition:

$$\begin{aligned}\left|h^F_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right| &\equiv \left|h^{(N,\epsilon_p)}_{\ell m}\hat{S}^{(\epsilon_p)}_{\rm eff}T_{\ell m}e^{i\delta_{\ell m}}f_{\ell m}(c_{\ell m})\right|\Big|_{t=t^{\ell m}_{\rm match}},\\ &= \left|h^{\rm NR}_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right|, \quad \text{for } (\ell,m)=(2,1),(5,5),(4,3),\end{aligned} \tag{34}$$

where $|h^{\rm NR}_{\ell m}\left(t^{\ell m}_{\rm match}\right)|$ is the amplitude of the NR modes at the matching point, given by fits in parameter space in Appendix A.

The remaining $N_{\ell m}$ term in Eq. (23) is the NQC correction and reads

$$\begin{aligned}N_{\ell m} = &\left[1+\frac{p^2_{r^*}}{(r\Omega)^2}\left(a^{h_{\ell m}}_1+\frac{a^{h_{\ell m}}_2}{r}+\frac{a^{h_{\ell m}}_3}{r^{3/2}}\right)\right]\\ &\times\exp\left[i\left(b^{h_{\ell m}}_1\frac{p_{r^*}}{r\Omega}+b^{h_{\ell m}}_2\frac{p^3_{r^*}}{r\Omega}\right)\right].\end{aligned} \tag{35}$$

The use of the NQC corrections guarantees that the modes' amplitude and frequency agree with NR fits, given in Appendix A, at the matching point $t^{\rm match}_{\ell m}$. In particular, one fixes the, the 5 constants $(a^{h_{\ell m}}_1, a^{h_{\ell m}}_2, a^{h_{\ell m}}_3, b^{h_{\ell m}}_1, b^{h_{\ell m}}_2)$ by requiring that [8, 9, 32]:

- The amplitude of the EOB modes is the same as that of the NR modes at the matching point $t^{\ell m}_{\rm match}$:

$$\left|h^{\rm insp\text{-}plunge}_{\ell m}(t^{\ell m}_{\rm match})\right| = \left|h^{\rm NR}_{\ell m}(t^{\ell m}_{\rm match})\right|. \tag{36}$$

  We notice that this condition is different from that in Eq. (34) because it affects $h^{\rm insp\text{-}plunge}_{\ell m}(t^{\ell m}_{\rm match})$ and not $h^{\rm F}_{\ell m}(t^{\ell m}_{\rm match})$. Because of the calibration parameter in Eq. (34), for the modes (2,1), (5,5) and (4,3) this condition becomes simply $|N_{\ell m}| = 1$.
- The first derivative of the amplitude of the EOB modes is the same as that of the NR modes at the matching point $t^{\ell m}_{\rm match}$:

$$\left.\frac{d\left|h^{\rm insp\text{-}plunge}_{\ell m}(t)\right|}{dt}\right|_{t=t^{\ell m}_{\rm match}} = \left.\frac{d\left|h^{\rm NR}_{\ell m}(t)\right|}{dt}\right|_{t=t^{\ell m}_{\rm match}}; \tag{37}$$

- The second derivative of the amplitude of the EOB modes is the same as that of the NR modes at the matching point $t^{\ell m}_{\rm match}$:

$$\left.\frac{d^2\left|h^{\rm insp\text{-}plunge}_{\ell m}(t)\right|}{dt^2}\right|_{t=t^{\ell m}_{\rm match}} = \left.\frac{d^2\left|h^{\rm NR}_{\ell m}(t)\right|}{dt^2}\right|_{t=t^{\ell m}_{\rm match}}; \tag{38}$$

- The frequency of the EOB modes is the same as that of the NR modes at the matching point $t^{\ell m}_{\rm match}$:

$$\omega^{\rm insp\text{-}plunge}_{\ell m}(t^{\ell m}_{\rm match}) = \omega^{\rm NR}_{\ell m}(t^{\ell m}_{\rm match}); \tag{39}$$

- The first derivative of the frequency of the EOB modes is the same as that of the NR modes at the matching point $t^{\ell m}_{\rm match}$:

$$\left.\frac{d\omega^{\rm insp\text{-}plunge}_{\ell m}(t)}{dt}\right|_{t=t^{\ell m}_{\rm match}} = \left.\frac{d\omega^{\rm NR}_{\ell m}(t)}{dt}\right|_{t=t^{\ell m}_{\rm match}}. \tag{40}$$

The RHS of Eqs. (36)–(40) (usually called *input values*), are given as fitting formulae for every point of the parameter space (ν,χ_1,χ_2) in Appendix A. These fits are produced using a catalog of 442 NR and 13 BH-perturbation-theory waveforms.

In `SEOBNRv5` the input values are enforced at $t = t^{\ell m}_{\rm match}$ given in Eq. (22) as function of $t^{22}_{\rm peak}$. We take

$$t^{22}_{\rm peak} = t_{\rm ISCO} + \Delta t_{22} \tag{41}$$

where $t_{\rm ISCO}$ is the time at which $r = r_{\rm ISCO}$, $r_{\rm ISCO}$ being computed from the final mass and spin of the remnant [33, 34], and Δt_{22} is a calibration parameter, to be determined by comparing against NR simulations. In `SEOBNRv4` the merger time was given by

$$t^{22}_{\rm peak} = t^{\Omega}_{\rm peak} + \Delta t^{22}_{\rm peak} \tag{42}$$

with $t^{\Omega}_{\rm peak}$ being the peak of the orbital frequency. The purpose of $\Delta t^{22}_{\rm peak}$ is still to introduce a time delay between the peak of the orbital frequency and the peak of the (2,2) mode, as observed in the test-particle limit [35–37]. However, we find the new definition to be more robust, since it is indipendent of features in the late dynamics, like the existence of a peak in the orbital frequency, which is not necessarily present for all BBH parameters when the Hamiltonian and radiation-reaction force are not the same of `SEOBNRv4`.

The EOB radiation-reaction force $\mathcal{F}$ is obtained by summing the amplitude of the factorized GW modes

$$\mathcal{F} \equiv \frac{\Omega}{16\pi}\frac{\boldsymbol{p}}{|\boldsymbol{L}|}\sum_{\ell=2}^{8}\sum_{m=-\ell}^{\ell} m^2\left|D_{\rm L}h^{\rm F}_{\ell m}\right|^2, \tag{43}$$

where Ω is the orbital frequency, and D_L is the luminosity distance of the binary to the observer. We point out that the NQC corrections are not included in the `SEOBNRv5` radiation-reaction, as well as the $c_{\ell m}$ calibration coefficients.

Main differences compared to `SEOBNRv4HM`:

- The high-order PN terms from Appendix A of [9] are now included in the RR force, and not just in the waveform modes.
- We add some of the terms recently derived in Ref. [38]. i) We add in ρ_{22} NLO spin-squared terms at 3PN and and LO spin-cubed terms at 3.5PN (Eq. (4.11a) from [38]). ii) We add all the known spin terms in the (3,2), (4,3) amplitudes (Eqs. (B2a) and (B5b) from [38]).

iii) We correct the expressions for the (2,1) mode. As pointed out in [38], the $O(v^6\chi^2\nu^2)$ terms in the (2,1) mode in SEOBNRv4HM [9] are not correct, as well as the $O(\nu v^5)$ nonspinning part of δ_{21}, whose coefficient had the value $-493/42$ [27, 30] instead of $-25/2$, due to an error in the (2,1) mode in Ref. [39], which was later corrected in an erratum. Since Ref. [38] was published only when the model was already close to being finalized, we only added the terms we considered most important, and we will add all new terms in a future update of the model. We remark that adding additional PN information in the waveform modes (except for the phases) modifies the flux, and would require a recalibration of the dynamics to NR.

- 2GSF calibration coefficients from [40]. In that work we define

$$\rho_{\ell m} = \rho_{\ell m}^{(0)} + \nu\rho_{\ell m}^{(1)} + O\left(\nu^2\right) \tag{44}$$

and augment the $\rho_{\ell m}^{(1),\mathrm{EOB}}$ by adding an additional polynomial $\Delta\rho_{\ell m}^{(1)}$ in v_Ω^2 starting at the lowest order in v_Ω^2 not already included. The $\Delta\rho_{\ell m}^{(1)}$ are determined by fitting to the numerical $\rho_{\ell m}^{(1),\mathrm{GSF}}$ results. The results of the fits are the following expressions:

$$\Delta\rho_{22}^{(1)} = 21.2v_\Omega^8 - 411v_\Omega^{10} \tag{45}$$

$$\Delta\rho_{21}^{(1)} = 1.65v_\Omega^6 + 26.5v_\Omega^8 + 80v_\Omega^{10} \tag{46}$$

$$\Delta\rho_{33}^{(1)} = 12v_\Omega^8 - 215v_\Omega^{10} \tag{47}$$

$$\Delta\rho_{32}^{(1)} = 0.333v_\Omega^6 - 6.5v_\Omega^8 + 98v_\Omega^{10} \tag{48}$$

$$\Delta\rho_{44}^{(1)} = -3.56v_\Omega^6 + 15.6v_\Omega^8 - 216v_\Omega^{10} \tag{49}$$

$$\Delta\rho_{43}^{(1)} = -0.654v_\Omega^4 - 3.69v_\Omega^6 + 18.5v_\Omega^8 \tag{50}$$

$$\Delta\rho_{55}^{(1)} = -2.61v_\Omega^4 + 1.25v_\Omega^6 - 35.7v_\Omega^8 \tag{51}$$

In [40] we also find it beneficial to include additional terms in the (3,2) and (4,3) mode obtained by matching to the PN expansions of the test particle flux.

$$\begin{aligned}\Delta\rho_{32}^{(1),\mathrm{TPL}} &= \frac{1}{\nu}\left(-\frac{1312549797426453052}{176264081083715625}\right.\\ &\quad\left.+\frac{18778864}{12629925}\,\mathrm{eulerlog}(2,v_\Omega)\right)v_\Omega^{10}\end{aligned} \tag{52}$$

$$\begin{aligned}\Delta\rho_{43}^{(1),\mathrm{TPL}} &= \frac{1}{\nu}\left(-\frac{2465107182496333}{460490801971200}\right.\\ &\quad\left.+\frac{174381}{67760}\,\mathrm{eulerlog}(3,v_\Omega)\right)v_\Omega^{8}\end{aligned} \tag{53}$$

where we define

$$\mathrm{eulerlog}\ (m,v_\Omega) \equiv \gamma + \log\left(2mv_\Omega\right) \tag{54}$$

in which γ is the Euler constant.

- We correct the coefficient of the $O(v^5\delta\chi_A\nu)$ term in ρ_{22}, whose value is 19/42, but was mistakenly replaced in the SEOBNRv4 code by 196/42.

The equations of motion for aligned spins, in terms of p_{r_*}, are given by Eqs. (10) of Ref. [27], and read

$$\begin{aligned}\dot{r} &= \xi\left.\frac{\partial H}{\partial p_{r_*}}\right|_r, \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi},\\ \dot{p}_{r_*} &= -\xi\left.\frac{\partial H}{\partial r}\right|_{p_{r_*}} + \frac{p_{r_*}}{p_\phi}\mathcal{F}_\phi, \qquad \dot{p}_\phi = \mathcal{F}_\phi.\end{aligned} \tag{55}$$

Quasicircular initial conditions are taken from [5]. One can then integrate numerically Eq. (55), to solve for the binary's dynamics.

In SEOBNRv5 one can also employ the post-adiabatic (PA) approximation for the inspiral dynamics, which allows to speed up the evaluation of the model, especially for very long waveforms [41–43]. The implementation of the PA dynamics closely follows that of Ref. [43] to which we refer for further details.

B. Merger-ringdown $h_{\ell m}$ modes

The merger-ringdown modes are constructed with a phenomenological ansatz, using information from numerical relativity (NR) simulations and test-particle limit (TPL) waveforms. The ansatz we employ for the modes (2,2), (3,3), (2,1), (4,4), (5,5), which show monotonic amplitude and frequency evolution, is the same as the one implemented in [8, 9] and reads:

$$h_{\ell m}^{\text{merger-RD}}(t) = \nu\tilde{A}_{\ell m}(t)e^{i\tilde{\phi}_{\ell m}(t)}e^{-i\sigma_{\ell m0}\left(t-t_{\mathrm{match}}^{\ell m}\right)} \tag{56}$$

where $\sigma_{\ell m0} = \sigma_{\ell m}^{\mathrm{R}} - i\sigma_{\ell m}^{\mathrm{I}}$ is the complex frequency of the least-damped QNM of the remnant BH. The QNM frequencies are obtained for each (ℓ,m) mode as a function of the BH's final mass and spin using the qnm python package [44]. The BH's mass and spin are in turn computed using the fitting formulas of [33] and [34] respectively. The ansätze for the two functions $\tilde{A}_{\ell m}$ $\tilde{\phi}_{\ell m}$ in Eq. (56) are the following [8, 9]

$$\tilde{A}_{\ell m}(t) = c_{1,c}^{\ell m}\tanh\left[c_{1,f}^{\ell m}\left(t-t_{\mathrm{match}}^{\ell m}\right)+c_{2,f}^{\ell m}\right]+c_{2,c}^{\ell m}, \tag{57}$$

$$\tilde{\phi}_{\ell m}(t) = \phi_{\mathrm{match}}^{\ell m} - d_{1,c}^{\ell m}\log\left[\frac{1+d_{2,f}^{\ell m}e^{-d_{1,f}^{\ell m}\left(t-t_{\mathrm{match}}^{\ell m}\right)}}{1+d_{2,f}^{\ell m}}\right], \tag{58}$$

where $\phi_{\mathrm{match}}^{\ell m}$ is the phase of the inspiral-plunge mode (ℓ,m) at $t = t_{\mathrm{match}}^{\ell m}$. The coefficients $d_{1,c}^{\ell m}$ and $c_{i,c}^{\ell m}$ $(i = 1,2)$ are constrained by the requirement that $\tilde{A}_{\ell m}(t)$ and $\tilde{\phi}_{\ell m}(t)$ in Eqs. (57), (58) are of class C^1 at $t = t_{\mathrm{match}}^{\ell m}$, and can be written in terms of $c_{1,f}^{\ell m}$, $c_{2,f}^{\ell m}$, $\sigma_{\ell m}^{\mathrm{R}}$, $\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\mathrm{match}}^{\ell m}\right)\right|$, $\partial_t\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\mathrm{match}}^{\ell m}\right)\right|$, as follows

$$\begin{aligned}c_{1,c}^{\ell m} &= \frac{1}{c_{1,f}^{\ell m}\nu}\left[\partial_t\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\mathrm{match}}^{\ell m}\right)\right|\right.\\ &\quad\left.-\sigma_{\ell m}^{\mathrm{R}}\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\mathrm{match}}^{\ell m}\right)\right|\right]\cosh^2\left(c_{2,f}^{\ell m}\right)\end{aligned} \tag{59}$$

$$c_{2,c}^{\ell m} = \frac{\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right|}{\nu} - \frac{1}{c_{1,f}^{\ell m}\nu}\left[\partial_t \left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right| - \sigma_{\ell m}^{\text{R}}\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right|\right] \cosh\left(c_{2,f}^{\ell m}\right) \sinh\left(c_{2,f}^{\ell m}\right), \tag{60}$$

or in terms of $d_{1,f}^{\ell m}, d_{2,f}^{\ell m}, \sigma_{\ell m}^{\text{I}}, \omega_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)$ for $d_{1,c}^{\ell m}$

$$d_{1,c}^{\ell m} = \left[\omega_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right) - \sigma_{\ell m}^{\text{I}}\right] \frac{1 + d_{2,f}^{\ell m}}{d_{1,f}^{\ell m} d_{2,f}^{\ell m}} \tag{61}$$

The NQC corrections in the inspiral-plunge modes make sure that the quantities $\left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right|$, $\partial_t \left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right|$, $\partial_t^2 \left|h_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)\right|$, $\omega_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)$, $\partial_t \omega_{\ell m}^{\text{insp-plunge}}\left(t_{\text{match}}^{\ell m}\right)$ coincide with the NR input values. The remaining parameters in Eqs. (57), (58) are the "free coefficients" $c_{i,f}^{\ell m}$ and $d_{i,f}^{\ell m}$, $i = 1, 2$. Using the NQCs allows to fit the free coefficients directly to NR, and makes the merger-ringdown modes indipendent of the EOB inspiral modes, allowing for a decoupled calibration of the two. To obtain these, we first extract them from each NR and TPL waveform by least-square fits, and then interpolate the values obtained across parameter space using polynomial fits in ν and χ. While in [9] the same polynomial was used for most of the free coefficients, in this work we used a recursive-feature-elimination (RFE) algorithm with polynomial features of third and fourth order, depending on the values to fit. Applying a log transformation to some of the coefficients was also found to be beneficial, both to improve the quality of the fits and to ensure positivity of those quantities when extrapolating outside of the region where NR data is available. A similar RFE strategy was also applied to most of the fits for the input values, the only exceptions being the fits of the amplitude of the odd-m modes (and their derivatives). The odd m modes vanish in the equal-mass and equal-spin limit, since they need to satisfy the symmetry under rotation $\phi_0 \to \phi_0 + \pi$, therefore the corresponding amplitudes are better captured by ad-hoc non-linear ansätze that enforce this limit by construction (see also Appendix B).

C. Mode mixing in the (3,2) and (4,3) modes

The merger-ringdown $(3,2), (4,3)$ modes show post-merger oscillations [45, 46], mostly related to the mismatch between the *spherical* harmonic basis used for extraction in NR simulations, and the *spheroidal* harmonics adapted to the perturbation theory of Kerr BHs. Because of this, it is not possible to use the same ansatz of Eqs. (56), (57), (58) straightforwardly.

Eq. (20) can be formulated in terms of -2 spin-weighted *spheroidal* harmonics as:

$$h(t; \boldsymbol{\theta}, \iota, \varphi_c) = \sum_{\ell' \geq 2} \sum_{|m| \leq \ell'} \sum_{n \geq 0} {}_{-2}S_{\ell' mn}(\iota, \varphi_c)\, {}^{S}h_{\ell mn}(t, \boldsymbol{\theta}), \tag{62}$$

where $S_{\ell mn} \equiv S_{\ell m}(a_f \sigma_{\ell mn})$ are the -2 spin-weighted spheroidal harmonics associated with the QNM frequencies $\sigma_{\ell mn}$, and with $a_f M_f$ being the spin angular momentum of the final BH of mass M_f [47]. The superscript S denotes that the ${}^{S}h_{\ell mn}$ modes are expanded in the spheroidal harmonics basis.

One can switch from the spherical harmonic basis to spheroidal harmonic basis via:

$$S_{\ell' mn}^{-2} = \sum_{\ell \geq |m|} \mu_{m\ell\ell' n}^{*} Y_{\ell m}^{-2}, \tag{63}$$

where $\mu_{m\ell\ell' n}$ are mode mixing coefficients, which we compute using fits provided by Berti and Klein [48] (more complicated fits can be found in [49]), and the star denotes the usual complex conjugation. Inserting Eq. (63) in Eq. (62) for the spheroidal harmonics we get,

$$h(t; \iota, \varphi_c) = \sum_{\ell' \geq 2} \sum_{|m| \leq \ell'} \sum_{n \geq 0} \sum_{\ell \geq |m|} \mu_{m\ell\ell' n}^{*} Y_{\ell m}^{-2}(\iota, \varphi_c)\, {}^{S}h_{\ell mn}(t), \tag{64}$$

where we have removed the $\boldsymbol{\theta}$ parameter from the expression to ease the notation. Comparing Eq. (64) with Eq. (20), we obtain the following relation between spherical and spheroidal modes,

$$h_{\ell m}(t) = \sum_{\ell' \geq |m|} \sum_{n \geq 0} {}^{S}h_{\ell' mn}(t) \mu_{m\ell\ell' n}^{*}. \tag{65}$$

Starting from Eq. (65), we can model the mode-mixing behavior [50] in such a way to obtain monotonic functions that can be fitted by the ansatz already used for the other modes. Practically, it is not feasible to sum over all the spheroidal modes to get each spherical mode, so we make a few plausible approximations: first, we neglect the overtone ($n > 0$) contributions in the right hand side of Eq. (65), because their decay times are $\gtrsim 3$ smaller than the dominant overtone $n = 0$. Second, for a given (ℓ, m) mode, we neglect the contributions from the spheroidal modes with $\ell' > \ell$ since their amplitudes are subdominant commpared to the $(\ell, m, 0)$ mode, and the corresponding mode mixing coefficients are also smaller. With these approximations, we can rewrite the Eq. (65) as

$$h_{\ell m}(t) \simeq \sum_{\ell' \leq \ell} {}^{S}h_{\ell' m0}(t) \mu_{m\ell\ell' 0}^{*}. \tag{66}$$

Writing it explicitly for the modes of interests,

$$h_{22}(t) \simeq \mu_{2220}^{*}\, {}^{S}h_{220}(t), \tag{67}$$
$$h_{33}(t) \simeq \mu_{3330}^{*}\, {}^{S}h_{330}(t), \tag{68}$$
$$h_{32}(t) \simeq \mu_{2320}^{*}\, {}^{S}h_{220}(t) + \mu_{2330}^{*}\, {}^{S}h_{320}(t), \tag{69}$$
$$h_{43}(t) \simeq \mu_{3430}^{*}\, {}^{S}h_{330}(t) + \mu_{3440}^{*}\, {}^{S}h_{430}(t). \tag{70}$$

From these equations, we can solve for the ${}^{S}h_{\ell m0}$ modes to obtain

$${}^{S}h_{320}(t) \simeq \frac{h_{32}(t) - h_{22}(t)\mu_{2320}^{*}/\mu_{2220}^{*}}{\mu_{2330}^{*}}, \tag{71}$$
$${}^{S}h_{430}(t) \simeq \frac{h_{43}(t) - h_{33}(t)\mu_{3430}^{*}/\mu_{3330}^{*}}{\mu_{3440}^{*}}. \tag{72}$$

The h_{32} mode shows oscillations in its amplitudes and frequency, while the ${}^{S}h_{320}$ mode obtained from Eq.(71) has a nearly monotonic behavior. Most importantly, the frequency of the ${}^{S}h_{320}$ mode oscillates around the QNM frequency predicted in BH perturbation theory for the spheroidal (3,2,0) mode.

From this reasoning it follows that we can model the spheroidal ${}^{S}h_{lm0}$ modes using the ansatz of Eq. (56), where in Eq. (58) $\phi^{\rm match}_{\ell m}$ is replaced by ${}^{S}\phi^{\rm match}_{\ell m0}$, the phase of ${}^{S}h_{lm0}$ at $t=t^{\rm match}_{\ell m}$, in Eq. (59), (60) $h_{\ell m}$ is replaced by ${}^{S}h_{lm0}$, and in Eq. (61) $\omega_{\ell m}$ by ${}^{S}\omega_{\ell m0}$. Once we have a model for ${}^{S}h_{320}$ and ${}^{S}h_{430}$, it is straightforward to obtain the (3,2) and (4,3) modes by combining them with the (2,2) and (3,3) ones previously obtained by inverting Eqs. (71), (72).

The NQC corrections for the inspiral-plunge $h_{\ell m}$ modes require the values at $t^{\rm match}_{\ell m}$ for the spherical NR modes $h^{\rm NR}_{\ell m}$, and those are the quantities that we fit and interpolate across parameter space. However, we need the input values of ${}^{S}h_{lm0}$ in order to fix the coefficients $c^{\ell m}_{i,c}$ and $d^{\ell m}_{i,c}$. They can be derived from the Eqs. (71) and (72) starting from the $h_{\ell m}$ input values. First, we introduce the following quantities,

$$\rho = |\mu_{m\ell\ell'0}|\frac{|h^{\rm match}_{\ell' m}|}{|\mu_{m\ell'\ell'0}||h^{\rm match}_{\ell m}|} \tag{73}$$

$$\delta\phi = \phi^{\ell' m}_{\rm match} - \phi^{\ell m}_{\rm match} - \arg(\mu_{m\ell\ell'0}) + \arg(\mu_{m\ell'\ell'0}) \tag{74}$$

$$F = \sqrt{(1-\rho\cos(\delta\phi))^2 + \rho^2\sin^2(\delta\phi)} \tag{75}$$

$$\alpha = \arctan\left(\frac{-\rho\sin(\delta\phi)}{1-\rho\cos(\delta\phi)}\right) \tag{76}$$

$$\dot\rho = |\mu_{m\ell\ell'0}|\left(\frac{\partial_t|h^{\rm match}_{\ell' m}|}{|h^{\rm match}_{\ell m}|} - \frac{|h^{\rm match}_{\ell' m}|}{|h^{\rm match}_{\ell m}|^2}\partial_t|h^{\rm match}_{\ell m}|\right) \tag{77}$$

$$\delta\dot\phi = \partial_t\phi^{\ell' m}_{\rm match} - \partial_t\phi^{\ell m}_{\rm match} \tag{78}$$

$$\dot F = (\rho\dot\rho + \rho\sin(\delta\phi)\delta\dot\phi - \dot\rho\cos(\delta\phi))/F \tag{79}$$

$$\dot\alpha = (\rho^2\delta\dot\phi - \rho\cos(\delta\phi)\delta\dot\phi - \dot\rho\sin(\phi))/F^2 \tag{80}$$

Where $|h^{\rm match}_{\ell m}| \equiv \left|h^{\text{insp-plunge}}_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right|$. Then,

$$|{}^{S}h^{\rm match}_{\ell m0}| = \frac{|h^{\rm match}_{\ell m}|F}{|\mu_{m\ell\ell 0}|} \tag{81}$$

$${}^{S}\phi^{\ell m0}_{\rm match} = \phi^{\ell m}_{\rm match} + \arg(\mu_{m\ell\ell 0}) + \alpha \tag{82}$$

$$\partial_t|{}^{S}h^{\rm match}_{\ell m0}| = \frac{(\partial_t|h^{\rm match}_{\ell m}|F + |h^{\rm match}_{\ell m}|\dot F)}{|\mu_{m\ell\ell 0}|} \tag{83}$$

$${}^{S}\omega^{\rm match}_{\ell m0} = \omega^{\rm match}_{\ell m} + \dot\alpha \tag{84}$$

where for (3,2) mode $m=2, \ell=3, \ell'=2$ and for (4,3) mode $m=3, \ell=4, \ell'=3$.

III. CALIBRATION TO NUMERICAL RELATIVITY

The inspiral-plunge modes described in Sec. II A are functions of the physical parameters (q,χ_1,χ_2), of the initial orbital frequency ω_0 at which the evolution is started, and of a set calibration parameters, that are determined as a function of (q,χ_1,χ_2) in order to maximize the agreement between the model and NR simulations. In `SEOBNRv5` we employ the following calibration parameters:

- a_6, a 5PN, linear in ν, parameter that enters the nonspinning $A_{\rm pm}(u)$ potential of Eqs. (9), (10).
- $d_{\rm SO}$, a 4.5PN spin-orbit parameter, that enters the odd-in-spin part of the effective Hamiltonian (see Eqs. (14), (15)).
- Δt_{22} a parameter that determines the time shift between the Kerr ISCO, computed from the final mass and spin of the remnant [33] [34], and the peak of the (2,2) mode amplitude (see Eq. (41)). We remark that this quantity is different from $\Delta t^{22}_{\rm peak}$ used in `SEOBNRv4`, where it corresponded to the time difference between the peak of the orbital frequency and the peak of the (2,2) mode amplitude.

As in `SEOBNRv4`, in `SEOBNRv5` we find it convenient to do the calibration in a hierarchical way, starting from nonspinning and then moving to aligned-spins. First, we calibrate to non-spinning configuration using as calibration parameters

$$\boldsymbol{\theta}_{\rm ns} \equiv \{a_6, \Delta t^{\rm ns}_{22}\}. \tag{85}$$

We then fix $a_6(\nu)$, $\Delta t^{\rm ns}_{22}(\nu)$ by the respective fits and calibrate to the remaining spin-aligned configurations using as calibration parameters

$$\boldsymbol{\theta}_{\rm s} \equiv \{d_{\rm SO}, \Delta t^{\rm s}_{22}\}, \tag{86}$$

where

$$\Delta t_{22} = \Delta t^{\rm ns}_{22} + \Delta t^{\rm s}_{22}, \tag{87}$$

and $\Delta t^{\rm s}_{22}$ is assumed to vanish in the non-spinning limit.

These are the calibration parameter fits that we obtain:

$$a_6 = 329523.262\nu^4 - 169019.14\nu^3 + 33414.4394\nu^2 - 3021.93382\nu + 41.787788 \tag{88}$$

$$\Delta t^{\rm ns}_{22} = \nu^{10.051322\nu - 1/5}\cdot\left(55565.2392\nu^3 - 9793.17619\nu^2 - 1056.87385\nu - 59.62318\right) \tag{89}$$

$$\Delta t^{\rm s}_{22} = \nu^{-1/5}\left(-6.789139a_+^4 + 5.399623a_+^3 + 6.389756a_+^2 a_- - 132.224951a_+^2\nu + 49.801644a_+^2\right.$$

$$+8.392389a_+a_-^2+179.569825a_+a_-\nu-40.606365a_+a_-+384.201019a_+\nu^2-141.253182a_+\nu$$
$$+17.571013a_+-16.905686a_-^2\nu+7.234106a_-^2+144.253396a_-\nu^2-90.192914a_-\nu+14.22031a_-\Big) \tag{90}$$

$$\begin{aligned}d_{\rm SO}=&-7.584581a_+^3-10.522544a_+^2a_--42.760113a_+^2\nu+18.178344a_+^2-17.229468a_+a_-^2\\&+362.767393a_+a_-\nu-85.803634a_+a_--201.905934a_+\nu^2-90.579008a_+\nu+49.629918a_+\\&-7.712512a_-^3-238.430383a_-^2\nu+69.546167a_-^2-1254.668459a_-\nu^2+472.431938a_-\nu\\&-39.742317a_-+478.546231\nu^3+679.52177\nu^2-177.334832\nu-37.689778.\end{aligned} \tag{91}$$

Appendix A: Fits of nonquasicircular input values

In this appendix we provide fits for the nonquasicircular (NQC) input values, $\left|h_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right|$, $\partial_t\left|h_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right|$, $\partial_t^2\left|h_{\ell m}\left(t^{\ell m}_{\rm match}\right)\right|$, $\omega_{\ell m}\left(t^{\ell m}_{\rm match}\right)$, $\partial_t\omega_{\ell m}\left(t^{\ell m}_{\rm match}\right)$. To produce the fits we used NR simulations with the highest level of resolution available and extrapolation order $N=2$. Depending on the mode, we excluded from the fits a different number of NR waveforms where numerical errors prevented us to fit them accurately. As in [9] we define the following combinations of m_1, m_2, χ_1, χ_2 to be used in the fits.

$$\delta=\frac{(m_1-m_2)}{(m_1+m_2)}), \tag{A1}$$
$$\chi_{33}=\chi_S\delta+\chi_A \tag{A2}$$
$$\chi_{21A}=\frac{\chi_S}{1-1.3\nu}\delta+\chi_A \tag{A3}$$
$$\chi_{44A}=(1-5\nu)\chi_S+\chi_A\delta \tag{A4}$$
$$\chi_{21D}=\frac{\chi_S}{1-2\nu}\delta+\chi_A \tag{A5}$$
$$\chi_{44D}=(1-7\nu)\chi_S+\chi_A\delta \tag{A6}$$
$$\chi=\chi_S+\chi_A\frac{\delta}{1-2\nu} \tag{A7}$$

The variables χ_{33}, χ_{21A}, χ_{21D} vanish by construction for equal-mass equal-spin configurations, and are used to enforce that the odd-m modes also vanish in the same limit as required by symmetry.

1. Amplitude's fits

$$\begin{aligned}\frac{|h^{\rm NR}_{22}(t^{\rm match}_{22})|}{\nu}=&|0.430147\chi^3\nu-0.084939\chi^3+0.619889\chi^2\nu^2-0.020826\chi^2-13.357614\chi\nu^3\\&+7.194264\chi\nu^2-1.743135\chi\nu+0.18694\chi+71.979698\nu^4-46.87586\nu^3\\&+12.440405\nu^2-0.868289\nu+1.467097|\end{aligned} \tag{A8}$$

$$\begin{aligned}\frac{|h^{\rm NR}_{33}(t^{\rm match}_{33})|}{\nu}=&|-0.088371\chi_{33}^2\delta\nu+0.036258\chi_{33}^2\delta+1.057731\chi_{33}\nu^2-0.466709\chi_{33}\nu\\&+0.099543\chi_{33}+1.96267\delta\nu^2+0.027833\delta\nu+0.558808\delta|\end{aligned} \tag{A9}$$

$$\begin{aligned}\frac{|h^{\rm NR}_{21}(t^{\rm match}_{21})|}{\nu}=&|-0.033175\chi_{21A}^3\delta+0.086356\chi_{21A}^2\delta\nu-0.049897\chi_{21A}^2\delta+0.012706\chi_{21A}\delta\\&+0.168668\chi_{21A}\nu-0.285597\chi_{21A}+1.067921\delta\nu^2-0.189346\delta\nu+0.431426\delta|\end{aligned} \tag{A10}$$

$$\begin{aligned}\frac{|h^{\rm NR}_{44}(t^{\rm match}_{44})|}{\nu}=&|0.031483\chi_{44A}^2-0.180165\chi_{44A}\nu+0.063931\chi_{44A}+6.239418\nu^3-1.947473\nu^2\\&-0.615307\nu+0.262533|\end{aligned} \tag{A11}$$

$$\begin{aligned}\frac{|h^{\rm NR}_{55}(t^{\rm match}_{55})|}{\nu}=&|-7.402839\chi_{33}\nu^3+3.965852\chi_{33}\nu^2-0.762776\chi_{33}\nu+0.062757\chi_{33}\\&+1.093812\delta\nu^2-0.462142\delta\nu+0.125468\delta|\end{aligned} \tag{A12}$$

$$\begin{aligned}\frac{|h^{\rm NR}_{32}(t^{\rm match}_{32})|}{\nu}=&|0.022598\chi^2+0.307803\chi\nu-0.020771\chi+8.917771\nu^3-2.194506\nu^2\\&-0.387911\nu+0.155446|\end{aligned} \tag{A13}$$

$$\frac{|h^{\rm NR}_{43}(t^{\rm match}_{43})|}{\nu} =|-0.071554\chi_{33}^2\delta\nu+0.021932\chi_{33}^2\delta-1.738079\chi_{33}\nu^2+0.436576\chi_{33}\nu -0.020081\chi_{33}+0.809615\delta\nu^2-0.273364\delta\nu+0.07442\delta| \tag{A14}$$

2. Amplitude-first-derivative's fits

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{22}(t)|}{dt}\bigg|_{t=t^{\rm match}_{22}} \equiv 0 \tag{A15}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{33}(t)|}{dt}\bigg|_{t=t^{\rm match}_{33}} =\chi_{33}^2\delta\,(0.004941\nu-0.002094) +0.001781\left|\chi_{33}^2+\chi_{33}\delta\,(39.247538\nu-2.986889)+\delta^2\,(85.173306\nu+4.637906)\right|^{1/2} \tag{A16}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{21}(t)|}{dt}\bigg|_{t=t^{\rm match}_{21}} =\chi_{21D}\delta\,(0.023534\nu-0.008064)+\delta\,(0.006743-0.0297\nu) +0.008256\left|\chi_{21D}-\delta\left(5.471011\nu^2+1.235589\nu+0.815482\right)\right| \tag{A17}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{44}(t)|}{dt}\bigg|_{t=t^{\rm match}_{44}} =-0.001251\chi_{44D}^3+0.006387\chi_{44D}^2\nu-0.001223\chi_{44D}^2-0.034308\chi_{44D}\nu^2 +0.014373\chi_{44D}\nu-0.000681\chi_{44D}+1.134679\nu^3-0.417056\nu^2 +0.024004\nu+0.003498 \tag{A18}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{55}(t)|}{dt}\bigg|_{t=t^{\rm match}_{55}} =\chi_{33}^2\delta\,(0.008568\nu-0.00155)+\chi_{33}\delta\,(0.002705\nu-0.001015) +\delta\,(0.002563-0.010891\nu)+0.000284\,|\chi_{33}+\delta\,(32.459725\nu+0.165336)| \tag{A19}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{32}(t)|}{dt}\bigg|_{t=t^{\rm match}_{32}} =-0.000806\chi^3-0.011027\chi^2\nu+0.002999\chi^2-0.14087\chi\nu^2+0.063211\chi\nu -0.006783\chi+1.693423\nu^3-0.510999\nu^2+0.020607\nu+0.003674 \tag{A20}$$

$$\frac{1}{\nu}\frac{d|h^{\rm NR}_{43}(t)|}{dt}\bigg|_{t=t^{\rm match}_{43}} =\chi_{33}^2\delta\,(0.001773-0.012159\nu)+\chi_{33}\delta\,(0.022249\nu-0.004295) +\delta\,(0.012043\nu-0.001067)+0.00082\,|\chi_{33}+\delta\,(3.880171-20.015436\nu)| \tag{A21}$$

3. Amplitude-second-derivative's fits

$$\frac{1}{\nu}\frac{d^2|h^{\rm NR}_{22}(t)|}{dt^2}\bigg|_{t=t^{\rm match}_{22}} =0.000386\chi^2+0.003589\chi\nu+0.001326\chi-0.003353\nu^2-0.005615\nu-0.002457 \tag{A22}$$

$$\frac{1}{\nu}\frac{d^2|h^{\rm NR}_{33}(t)|}{dt^2}\bigg|_{t=t^{\rm match}_{33}} =\chi_{33}\delta\,(0.000552\nu+0.001029)-0.000218 \cdot\left|\chi_{33}+\delta\left(-2188.340923\nu^4+1331.981345\nu^3-289.772357\nu^2+32.212775\nu+3.396168\right)\right| \tag{A23}$$

$$\frac{1}{\nu}\frac{d^2|h^{\rm NR}_{21}(t)|}{dt^2}\bigg|_{t=t^{\rm match}_{21}} =0.00015\delta-\Big|0.000316\chi_{21D}^3-\chi_{21D}^2\delta\left(-0.043291\nu^2+0.005682\nu+0.000502\right)$$

$$+0.000372\chi_{21D}\delta-\delta\Big(0.003643\nu+2.8\cdot10^{-5}\Big)\Big| \tag{A24}$$

$$\begin{aligned}\frac{1}{\nu}\frac{d^2|h_{44}^{\rm NR}(t)|}{dt^2}\bigg|_{t=t_{44}^{\rm match}} =& -0.000591\chi^2\nu+0.000174\chi^2-0.000501\chi\nu+0.000318\chi+0.138496\nu^3\\ &-0.047008\nu^2+0.003899\nu-0.000451\end{aligned} \tag{A25}$$

$$\frac{1}{\nu}\frac{d^2|h_{55}^{\rm NR}(t)|}{dt^2}\bigg|_{t=t_{55}^{\rm match}} =\chi_{33}^2\cdot\Big(0.000278\nu-5.6\cdot10^{-5}\Big)+\chi_{33}\delta\Big(0.000246\nu-6.8\cdot10^{-5}\Big)+\delta\Big(0.000118-5.9\cdot10^{-5}\nu\Big) \tag{A26}$$

$$\begin{aligned}\frac{1}{\nu}\frac{d^2|h_{32}^{\rm NR}(t)|}{dt^2}\bigg|_{t=t_{32}^{\rm match}} =& -0.002882\chi^2\nu+0.000707\chi^2-0.027461\chi\nu^2+0.008481\chi\nu-0.000691\chi\\ &+0.20836\nu^3-0.053191\nu^2+0.001604\nu-5.6\cdot10^{-5}\end{aligned} \tag{A27}$$

$$\begin{aligned}\frac{1}{\nu}\frac{d^2|h_{43}^{\rm NR}(t)|}{dt^2}\bigg|_{t=t_{43}^{\rm match}} =& \chi_{33}\delta(0.00291\nu-0.000348)-5.0\cdot10^{-6}\\ &\cdot\Big|\chi_{33}+\delta\Big(-25646.358742\nu^4+12647.805787\nu^3+291.751053\nu^2-531.965263\nu+23.849357\Big)\Big|\end{aligned} \tag{A28}$$

4. Frequency and frequency-derivative fits

$$\begin{aligned}\omega_{22}^{\rm NR}(t_{22}^{\rm match}) =& -0.015259\chi^4+0.241948\chi^3\nu-0.066927\chi^3-0.971409\chi^2\nu^2+0.518014\chi^2\nu\\ &-0.087152\chi^2+3.751456\chi\nu^3-1.697343\chi\nu^2+0.250965\chi\nu-0.091339\chi\\ &+5.893523\nu^4-3.349305\nu^3+0.285392\nu^2-0.317096\nu-0.268541\end{aligned} \tag{A29}$$

$$\begin{aligned}\omega_{33}^{\rm NR}(t_{33}^{\rm match}) =& -0.045141\chi^3+0.346675\chi^2\nu-0.119419\chi^2-0.745924\chi\nu^2+0.478915\chi\nu\\ &-0.17467\chi+8.887163\nu^3-4.226831\nu^2-0.427167\end{aligned} \tag{A30}$$

$$\begin{aligned}\omega_{21}^{\rm NR}(t_{21}^{\rm match}) =& -0.01009\chi^3+0.077343\chi^2\nu-0.02411\chi^2-0.168854\chi\nu^2+0.159382\chi\nu\\ &-0.047635\chi-1.965157\nu^3+0.53085\nu^2-0.237904\nu-0.176526\end{aligned} \tag{A31}$$

$$\begin{aligned}\omega_{44}^{\rm NR}(t_{44}^{\rm match}) =& -0.042529\chi^3+0.415864\chi^2\nu-0.155222\chi^2-0.768712\chi\nu^2+0.592568\chi\nu\\ &-0.244508\chi+13.651335\nu^3-5.490329\nu^2-0.574041\end{aligned} \tag{A32}$$

$$\begin{aligned}\omega_{55}^{\rm NR}(t_{55}^{\rm match}) =& -0.091629\chi^3+0.802759\chi^2\nu-0.246646\chi^2-3.04576\chi\nu^2+1.43471\chi\nu\\ &-0.329591\chi+13.81386\nu^3-6.61611\nu^2+0.472474\nu-0.589341\end{aligned} \tag{A33}$$

$$\begin{aligned}\omega_{32}^{\rm NR}(t_{32}^{\rm match}) =& -0.045647\chi^2-2.758635\chi\nu^2+0.811353\chi\nu-0.112477\chi-2.346024\nu^3\\ &+1.57986\nu^2-0.317756\nu-0.331141\end{aligned} \tag{A34}$$

$$\begin{aligned}\omega_{43}^{\rm NR}(t_{43}^{\rm match}) =& -0.037919\chi^3+0.226903\chi^2\nu-0.087288\chi^2-0.905919\chi\nu^2+0.291092\chi\nu\\ &-0.1198\chi-55.534105\nu^3+23.913277\nu^2-3.487986\nu-0.34306\end{aligned} \tag{A35}$$

$$\begin{aligned}\dot\omega_{22}^{\rm NR}(t_{22}^{\rm match}) =& 0.000614\chi^3-0.008393\chi^2\nu+0.001948\chi^2+0.07799\chi\nu^2-0.028772\chi\nu\\ &+0.001705\chi-0.237126\nu^3+0.092215\nu^2-0.03104\nu-0.005484\end{aligned} \tag{A36}$$

$$\begin{aligned}\dot\omega_{33}^{\rm NR}(t_{33}^{\rm match}) =& 0.001697\chi^3-0.016231\chi^2\nu+0.003985\chi^2+0.154378\chi\nu^2-0.050618\chi\nu\\ &+0.002721\chi+0.255402\nu^3-0.08663\nu^2-0.027405\nu-0.009736\end{aligned} \tag{A37}$$

$$\begin{aligned}\dot\omega_{21}^{\rm NR}(t_{21}^{\rm match}) =& 0.00149\chi^3-0.008965\chi^2\nu+0.002739\chi^2+0.033831\chi\nu^2-0.005752\chi\nu\\ &+0.002003\chi-0.204368\nu^3+0.120705\nu^2-0.035144\nu-0.006579\end{aligned} \tag{A38}$$

$$\begin{aligned}\dot\omega_{44}^{\rm NR}(t_{44}^{\rm match}) =& 0.001812\chi^3-0.024687\chi^2\nu+0.00568\chi^2+0.162693\chi\nu^2-0.061205\chi\nu\\ &+0.003623\chi+0.536664\nu^3-0.094797\nu^2-0.045406\nu-0.013038\end{aligned} \tag{A39}$$

$$\dot{\omega}^{\rm NR}_{55}(t^{\rm match}_{55}) = 0.001509\chi^3 - 0.01547\chi^2\nu + 0.002802\chi^2 + 0.164011\chi\nu^2 - 0.056516\chi\nu \\ + 0.002072\chi + 0.043963\nu^3 + 0.048045\nu^2 - 0.045197\nu - 0.008688 \tag{A40}$$

$$\dot{\omega}^{\rm NR}_{32}(t^{\rm match}_{32}) = -0.036711\chi^2\nu + 0.005532\chi^2 + 0.09192\chi\nu^2 - 0.030713\chi\nu + 0.005927\chi \\ - 2.494788\nu^3 + 0.995116\nu^2 - 0.10163\nu - 0.010763 \tag{A41}$$

$$\dot{\omega}^{\rm NR}_{43}(t^{\rm match}_{43}) = 0.000537\chi^3 - 0.009876\chi^2\nu + 0.003279\chi^2 + 0.13296\chi\nu^2 - 0.060884\chi\nu \\ + 0.008513\chi - 5.160613\nu^3 + 2.180781\nu^2 - 0.292607\nu - 0.005308 \tag{A42}$$

Appendix B: Fits for amplitude and phase of merger-ringdown model

In this appendix we provide fits across parameter space for the free coefficients in the merger-ringdown anstaz given by Eqs. (57), (58). To produce the fits we used NR simulations with the highest level of resolution available and extrapolation order $N = 2$.

$$c^{22}_{1,f} = -0.001777\chi^4 + 0.062842\chi^3\nu - 0.018908\chi^3 + 0.013161\chi^2\nu^2 + 0.049388\chi^2\nu \\ - 0.019314\chi^2 + 1.867978\chi\nu^3 - 0.702488\chi\nu^2 + 0.033885\chi\nu - 0.011612\chi \\ - 4.238246\nu^4 + 2.043712\nu^3 - 0.406992\nu^2 + 0.053589\nu + 0.086254 \tag{B1}$$

$$c^{22}_{2,f} = 1.021875\chi^3\nu - 0.20348\chi^3 - 3.556173\chi^2\nu^2 + 1.970082\chi^2\nu - 0.264297\chi^2 \\ + 2.002947\chi\nu^3 - 5.585851\chi\nu^2 + 1.837724\chi\nu - 0.27076\chi - 63.286459\nu^4 \\ + 44.331389\nu^3 - 9.529573\nu^2 + 1.155695\nu - 0.528763 \tag{B2}$$

$$d^{22}_{1,f} = -0.013321\chi^4 + 0.047305\chi^3\nu - 0.024203\chi^3 + 1.033352\chi^2\nu^2 - 0.254351\chi^2\nu \\ - 0.007847\chi^2 + 4.113463\chi\nu^3 - 1.652924\chi\nu^2 + 0.090834\chi\nu - 28.423701\nu^4 \\ + 20.719874\nu^3 - 6.075679\nu^2 + 0.780093\nu + 0.135758 \tag{B3}$$

$$d^{22}_{2,f} = \exp(-0.163113\chi^4 - 3.398858\chi^3\nu + 0.728816\chi^3 + 23.975132\chi^2\nu^2 - 10.064954\chi^2\nu \\ + 1.2115\chi^2 + 9.057306\chi\nu^3 - 5.268296\chi\nu^2 + 0.464553\chi\nu + 0.56269\chi \\ - 352.249383\nu^4 + 275.843499\nu^3 - 81.483314\nu^2 + 11.184576\nu + 0.03571) \tag{B4}$$

$$c^{33}_{1,f} = -0.00956\chi^3 + 0.029459\chi^2\nu - 0.020264\chi^2 - 0.494524\chi\nu^2 + 0.169463\chi\nu \\ - 0.026285\chi - 5.847417\nu^3 + 1.957462\nu^2 - 0.171682\nu + 0.093539 \tag{B5}$$

$$c^{33}_{2,f} = -0.057346\chi^3 + 0.237107\chi^2\nu - 0.094285\chi^2 - 4.250609\chi\nu^2 + 1.763105\chi\nu \\ - 0.315826\chi + 14.801916\nu^3 - 7.060581\nu^2 + 1.158627\nu - 0.646888 \tag{B6}$$

$$d^{33}_{1,f} = -0.016524\chi^3 + 0.221466\chi^2\nu - 0.066323\chi^2 + 0.678442\chi\nu^2 - 0.261264\chi\nu \\ + 0.006664\chi + 2.316434\nu^3 - 2.192227\nu^2 + 0.424582\nu + 0.161577 \tag{B7}$$

$$d^{33}_{2,f} = \exp(0.275999\chi^3 - 1.830695\chi^2\nu + 0.512734\chi^2 + 29.072515\chi\nu^2 - 10.581319\chi\nu \\ + 1.310643\chi + 324.310223\nu^3 - 124.681881\nu^2 + 13.200426\nu + 0.410855) \tag{B8}$$

$$c^{21}_{1,f} = 0.173462\chi^2\nu - 0.028873\chi^2 + 0.197467\chi\nu^2 - 0.026139\chi - 2.934735\nu^3 \\ + 1.009106\nu^2 - 0.112721\nu + 0.099889 \tag{B9}$$

$$c^{21}_{2,f} = 0.183489\chi^3 + 0.10573\chi^2 - 20.792825\chi\nu^2 + 6.867746\chi\nu - 0.484948\chi \\ - 54.917585\nu^3 + 16.466312\nu^2 + 0.426316\nu - 0.92208 \tag{B10}$$

$$d^{21}_{1,f} = 0.018467\chi^4 + 0.398621\chi^3\nu - 0.050499\chi^3 - 0.877201\chi^2\nu^2 + 0.414553\chi^2\nu \\ - 0.068277\chi^2 - 10.648526\chi\nu^3 + 4.104918\chi\nu^2 - 0.723576\chi\nu + 0.039227\chi$$

$$+42.715534\nu^4 - 18.280603\nu^3 + 2.236592\nu^2 - 0.048094\nu + 0.16335 \quad \text{(B11)}$$

$$d_{2,f}^{21} = \exp(0.814085\chi^3 - 1.197363\chi^2\nu + 0.560622\chi^2 + 6.44667\chi\nu^2 - 5.630563\chi\nu + 0.949586\chi + 91.269183\nu^3 - 27.329751\nu^2 + 1.101262\nu + 1.040761) \quad \text{(B12)}$$

$$c_{1,f}^{44} = 4.519504\chi\nu^2 - 1.489036\chi\nu + 0.068403\chi - 1656.065439\nu^4 + 817.835726\nu^3 - 127.055379\nu^2 + 6.921968\nu + 0.009386 \quad \text{(B13)}$$

$$c_{2,f}^{44} = 0.964861\chi^3\nu - 0.185226\chi^3 - 12.647814\chi^2\nu^2 + 5.264969\chi^2\nu - 0.539721\chi^2 - 254.719552\chi\nu^3 + 105.698791\chi\nu^2 - 12.107281\chi\nu + 0.2244\chi - 393.727702\nu^4 + 145.32788\nu^3 - 15.556222\nu^2 + 1.592449\nu - 0.677664 \quad \text{(B14)}$$

$$d_{1,f}^{44} = -0.020644\chi^3 + 0.494221\chi^2\nu - 0.127074\chi^2 + 4.297985\chi\nu^2 - 1.284386\chi\nu + 0.062684\chi - 44.280815\nu^3 + 11.021482\nu^2 - 0.162943\nu + 0.166018 \quad \text{(B15)}$$

$$d_{2,f}^{44} = \exp(37.735116\chi\nu^2 - 12.516669\chi\nu + 1.309868\chi - 528.368915\nu^3 + 155.115196\nu^2 - 6.612448\nu + 0.787726) \quad \text{(B16)}$$

$$c_{1,f}^{55} = -0.009957\chi^3 + 0.059748\chi^2\nu - 0.02146\chi^2 - 0.206811\chi\nu^2 + 0.055078\chi\nu - 0.014528\chi - 5.966891\nu^3 + 1.76928\nu^2 - 0.055272\nu + 0.080368 \quad \text{(B17)}$$

$$c_{2,f}^{55} = 0.119703\chi^4 + 1.638345\chi^2\nu^2 - 0.064725\chi^2 - 28.499278\chi\nu^3 + 3.73034\chi\nu^2 + 1.853723\chi\nu - 0.225283\chi - 1887.591102\nu^4 + 794.134711\nu^3 - 107.010824\nu^2 + 6.32117\nu - 1.507483 \quad \text{(B18)}$$

$$d_{1,f}^{55} = -0.021537\chi^3 + 0.168071\chi^2\nu - 0.050263\chi^2 + 0.871799\chi\nu^2 - 0.230057\chi\nu + 9.018546\nu^3 - 5.009488\nu^2 + 0.606313\nu + 0.150622 \quad \text{(B19)}$$

$$d_{2,f}^{55} = \exp(28.839035\chi\nu^2 - 9.726025\chi\nu + 0.901423\chi + 143.745208\nu^3 - 64.478227\nu^2 + 6.223833\nu + 2.058139) \quad \text{(B20)}$$

$$c_{1,f}^{32} = -0.133035\chi^3 + 0.641681\chi^2\nu - 0.111865\chi^2 + 8.987763\chi\nu^2 - 1.582259\chi\nu + 0.095604\chi - 26.991806\nu^3 + 13.716801\nu^2 - 1.63083\nu + 0.157543 \quad \text{(B21)}$$

$$c_{2,f}^{32} = 0.121608\chi^3 - 1.590623\chi^2\nu + 0.167231\chi^2 - 25.544931\chi\nu^2 + 10.127968\chi\nu - 0.999062\chi - 51.469773\nu^3 + 46.209833\nu^2 - 6.484571\nu - 0.716883 \quad \text{(B22)}$$

$$d_{1,f}^{32} = \exp(-0.764015\chi^3 - 8.684722\chi^2\nu + 0.691946\chi^2 - 0.518291\chi\nu^2 - 1.407934\chi\nu + 0.236427\chi + 81.222175\nu^3 - 18.040529\nu^2 + 2.216406\nu - 1.879455) \quad \text{(B23)}$$

$$d_{2,f}^{32} = \exp(-1.819822\chi^3 - 24.501503\chi^2\nu + 3.287882\chi^2 - 39.324579\chi\nu^2 + 14.379901\chi\nu - 215.372399\nu^3 + 136.20936\nu^2 - 16.842816\nu + 1.463485) \quad \text{(B24)}$$

$$c_{1,f}^{43} = 0.041585\chi^3 + 4.188908\chi\nu^2 - 1.365732\chi\nu + 0.058908\chi + 44.311948\nu^3 - 22.114177\nu^2 + 3.386082\nu - 0.035315 \quad \text{(B25)}$$

$$c_{2,f}^{43} = 0.125764\chi^3 + 0.337235\chi^2\nu + 0.146202\chi^2 - 9.803187\chi\nu^2 + 3.995199\chi\nu - 0.240976\chi - 57.968821\nu^3 + 7.820929\nu^2 + 3.364741\nu - 1.121716 \quad \text{(B26)}$$

$$d_{1,f}^{43} = \exp(-0.888286\chi^3 + 3.97869\chi^2\nu - 1.047181\chi^2 - 14.823391\chi\nu^2 + 6.940856\chi\nu - 0.367801\chi + 366.645645\nu^3 - 161.732513\nu^2 + 19.564699\nu - 2.29578) \quad \text{(B27)}$$

$$d_{2,f}^{43} = \exp(-0.950676\chi^3 - 0.31428\chi^2 + 39.21796\chi\nu^2 - 10.651167\chi\nu + 1.339732\chi + 730.42296\nu^3 - 312.960598\nu^2 + 37.402567\nu - 0.061894) \quad \text{(B28)}$$

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