

# Pcal beam displacement investigation methodology

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**Abstract.** Calculation of expected variations in the Pcal X/Y comparison when one Pcal beam is moved by a known amount. This topic was originally investigated in collaboration with Julianna Lewis during her SURF project at LHO in the summer of 2023. Expected relative changes in the X/Y comparison for the LHO X-end ETM when one Pcal beam is moved horizontally and vertically by 5 mm at the Rx power sensor aperture are calculated.

## 1. Pcal-induced test mass displacements

We start with equation 2 in [LIGO-P2000113](#) that gives the displacement *sensed* by the interferometer resulting from both induced periodic longitudinal displacement and induced periodic rotation of the ETM.

$$x(\omega) = \frac{2 \cos \theta}{c} P(\omega) [S(\omega) + R(\omega)(\vec{a} \cdot \vec{b})] \simeq -\frac{2 \cos \theta}{Mc\omega^2} P(\omega) \left[ 1 + \frac{M}{I} (\vec{a} \cdot \vec{b}) \right]. \quad (1)$$

Here  $\vec{b}$  is displacement vector for the location of the interferometer beam away from the center of the face of the optic.  $\vec{a}$  is the displacement vector of the position of the center of force for the two Pcal beams on the ETM surface given by

$$\vec{a} = \frac{\vec{a}_1 + \vec{a}_2}{2} \quad (2)$$

where  $\vec{a}_1$  and  $\vec{a}_2$  are the displacement vectors for the positions of the two Pcal beams. For simplicity, throughout this note we assume that the powers in the two Pcal beams are equal. Otherwise the displacement vector for each beam would be weighted by the fraction of the power in the beam. The second term in square brackets on the right side of (1) represents the *apparent* displacement resulting from rotation of the ETM induced by mis-located ( $\vec{a} \neq 0$ ) Pcal beams that would be sensed by the interferometer, but not accounted for by the signal from the Pcal Rx sensor. It is treated as a Type B uncertainty in the Pcal error budget.

## 2. Pcal X/Y comparison

The Pcal X/Y calibration comparison uses the interferometer output signals to compare the calibrations of the X-end and Y-end Pcal systems, leveraging both the inherent insensitivity (except for sign) of the LIGO interferometers to which ETM is moving and the observed stability of the X/Y comparison calculated during the O4 observing run (see [LIGO-P2300412](#)) to enable investigation of changes as small as a few parts in 10,000 (a few hops). The X/Y comparison calculation compares the amplitudes of Pcal-induced modulations in the calibrated Pcal Rx sensor output signals to those in an interferometer output signal. This interferometer signal does not need to be well calibrated per se. Modeling indicates that the interferometer signals are insensitive at the ppm level to which arm is moving. However, any differences in the response of the signal used to length variations over the 0.1 Hz span of the two Pcal excitations must be taken into account. In this note, we assume that we are using an interferometer signal, denoted by  $I_{sig}$ , such as the GDS\_CALIB\_STRAIN signal that *reconstructs* the *external* displacement that is suppressed by the DARM loop<sup>†</sup> The Pcal X/Y calibration comparison factor is thus given by

$$\chi_{XY} = \frac{x(\omega_X)}{I_{sig}(\omega_X)/\mathcal{R}(\omega_X)} \left[ \frac{x(\omega_Y)}{I_{sig}(\omega_Y)/\mathcal{R}(\omega_Y)} \right]^{-1} = \frac{x(\omega_X)}{x(\omega_Y)} \frac{I_{sig}(\omega_Y)}{I_{sig}(\omega_X)} \frac{1}{\mathcal{R}_{XY}}, \quad (3)$$

where  $x(\omega_x)$  and  $x(\omega_y)$  are the amplitudes of the modulated displacements in the calibrated (in m) outputs of the Pcal X-end and Y-end Rx sensors at their respective modulation frequencies,  $\omega_x$  and  $\omega_y$ ,  $I_{sig}(\omega_x)$  and  $I_{sig}(\omega_y)$  are the amplitudes of the corresponding modulations in the interferometer output signal, and  $\mathcal{R}_{XY} = \mathcal{R}(\omega_Y)/\mathcal{R}(\omega_X)$  is the length response ratio for the interferometer output signal.

The ratio of the calibrated Pcal displacement amplitudes is given by

$$\frac{x(\omega_X)}{x(\omega_Y)} = \frac{Rx_X(\omega_X) X_X/\omega_X^2}{Rx_Y(\omega_Y) X_Y/\omega_Y^2} \quad (4)$$

where  $Rx_X$  and  $Rx_Y$  are the raw, digital Pcal Rx sensor outputs (e.g. H1:CAL-PCALX\_RX\_PD\_OUT\_DQ) and  $X_X$  and  $X_Y$  are the *independent* Pcal displacement factors.

Following the formalism in section 2.2 in [LIGO-P2000113](#)), we use the measured value of  $\chi_{XY}$  to calculate combined calibration correction factors  $C_X$  and  $C_Y$ . The *corrected* Pcal X/Y calibration comparison factor is then given by

$$\chi_{XY}^c = \frac{Rx_X(\omega_X) X_X^c/\omega_x^2}{Rx_Y(\omega_Y) X_Y^c/\omega_y^2} \frac{I_{sig}(\omega_Y)}{I_{sig}(\omega_X)} \frac{1}{\mathcal{R}_{XY}} = \frac{\chi_{XY}}{C_X C_Y}. \quad (5)$$

<sup>†</sup> Note that if, instead, the DARM\_ERR signal were used, as it is in the front-end code, the DARM loop response function ratio at the two Pcal frequencies,  $R_{XY}$ , would have to be taken into account as detailed in [LIGO-T2400108](#). In this note we assume that the reconstruction involved in generating  $I_{sig}$  treats displacements induced at the two Pcal excitation frequencies, separated by 0.1 Hz, equally.

It should initially be equal to unity at the time the calibration correction factors are calculated and subsequent variations from unity are expected to indicate changes in the ratio of the Pcal end station calibrations.

At the beginning of the O4 observing run, before applying correction factors in the front-end code, the X/Y comparison factor  $\chi_{XY}^0$  was measured and  $C_X$  and  $C_Y$  were calculated as detailed in [LIGO-P2000113](#). They were applied in the front-end code such that  $\chi_{XY}^c = 1/R_{XY}$ . At the start of the start of the O4b observing run,  $R_{XY}$  was 1.00535 for H1 and 0.999456 for L1 (see [LIGO-T2400108](#)).

### 3. Impact of unintended rotation of the ETM due to Pcal forces

The Pcal-induced displacements that are sensed by the interferometer include the *apparent* longitudinal displacement induced by unintended rotation of the ETM caused by unbalanced Pcal forces. This imbalance results from either unequal powers in the two Pcal beams or misplacement of the Pcal beams on the ETM surface resulting in a net periodic torque. However, the Pcal Rx sensors are NOT sensitive to the apparent displacements resulting from ETM rotation. Thus,

$$\chi_{XY}^c \propto \frac{\left[1 + \frac{M_Y}{I_Y}(\vec{a} \cdot \vec{b})\right]_{Y\text{-end}}}{\left[1 + \frac{M_X}{I_X}(\vec{a} \cdot \vec{b})\right]_{X\text{-end}}}. \quad (6)$$

It was equal to unity at the beginning of the O4 run when any initial displacements of the composite center of force of the Pcal beams ( $\vec{a}$ ) and/or the interferometer beam ( $\vec{b}$ ) were incorporated into the X-arm and Y-arm correction factors,  $C_X$  and  $C_Y$ .

### 4. Intentional displacement of one Pcal beam

If we were to intentionally displace one of the Pcal beams at the X-end station by a known amount by, for instance, using an aperture at the Rx sensor (see [LHO aLog 72063](#))<sup>‡</sup>, the Pcal center of force displacement vector would become

$$\vec{a}' = \frac{\vec{a}_1 + \vec{a}_2 + \vec{c}}{2} = \vec{a} + \frac{\vec{c}}{2} \quad (7)$$

where  $\vec{c}$  is the intentional displacement of one of the beams. (As noted earlier, we assume that the powers in the two beams are equal.) Thus the relative change in the

<sup>‡</sup> Because the ETM is approximately halfway between the Tx module and the Rx module, the displacement on the ETM surface is about half of the displacement induced at the Rx module for a position change induced by a mirror adjustment in the Rx module.

X/Y comparison factor due to the displacement of the Pcal beam would be given by

$$\begin{aligned} \frac{\chi'_{XY}}{\chi^0_{XY}} &= \frac{\left[1 + \frac{M_Y}{I_Y}(\vec{a} \cdot \vec{b})\right]_{Y\text{-end}}}{\left[1 + \frac{M_X}{I_X}(\vec{a} \cdot \vec{b})\right]_{X\text{-end}}} \times \frac{\left[1 + \frac{M_X}{I_X}(\vec{a} \cdot \vec{b})\right]_{X\text{-end}}}{\left[1 + \frac{M_Y}{I_Y}(\vec{a} \cdot \vec{b})\right]_{Y\text{-end}}} \\ &= \frac{\left[1 + \frac{M_X}{I_X}(\vec{a} \cdot \vec{b})\right]_{X\text{-end}}}{\left[1 + \frac{M_X}{I_X}(\vec{a} \cdot \vec{b}) + \frac{M_X}{I_X}\left(\frac{\vec{c}}{2} \cdot \vec{b}\right)\right]_{X\text{-end}}} = \frac{1 + \alpha}{1 + \alpha + \beta} \end{aligned} \quad (8)$$

where

$$\alpha = \frac{M_X}{I_X}(\vec{a} \cdot \vec{b})_{X\text{-end}} \quad \text{and} \quad \beta = \frac{M_X}{I_X}\left(\frac{\vec{c}}{2} \cdot \vec{b}\right)_{X\text{-end}}. \quad (9)$$

For the aLIGO ETMs  $M/I$  is about  $0.94 \times 10^{-4}/\text{mm}^2$  and both  $|\vec{a}|$  and  $|\vec{c}|/2$  are 1-2 mm or less. At LHO,  $|\vec{b}|$  is as large as 29 mm due to point absorbers in the ETM coating; at LLO it is an order of magnitude or more smaller. Thus both  $\alpha$  and  $\beta$  are  $\leq 0.006$ , much smaller than 1, and (8) can be approximated as

$$\frac{\chi'_{XY}}{\chi^0_{XY}} \simeq (1 + \alpha)(1 - \alpha - \beta) = 1 - \beta - \alpha^2 - \alpha\beta. \quad (10)$$

Ignoring terms that are second order in  $\alpha$  and  $\beta$ , we have

$$\frac{\chi'_{XY}}{\chi^0_{XY}} \simeq 1 - \beta = 1 - \frac{M_X}{I_X}\left(\frac{\vec{c}}{2} \cdot \vec{b}\right)_{X\text{-end}} = 1 - \frac{M_X}{2I_X}\left(\frac{c_i}{2}b_i + \frac{c_j}{2}b_j\right)_{X\text{-end}}. \quad (11)$$

## 5. Relative change expected from Pcal beam displacements in August 2023

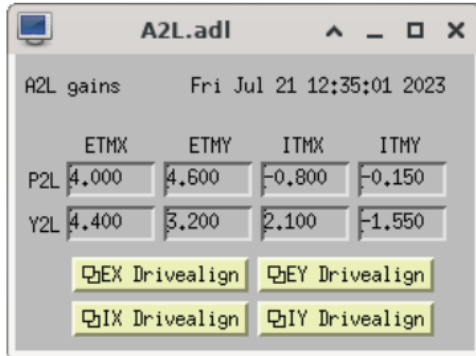
For the experiment carried out in August 2023 by Rick Savage, Tony Sanchez, and Julianna Lewis (see [LHO aLog 72063](#)), one Pcal beam (the inner, or upper beam) was moved by 5 mm at the Rx sensor input aperture which would correspond to about 2.5 mm at the ETM surface. First, the beam was moved down, then it was moved to the left.

Figure 1 shows an excerpt from an email from J. Driggers indicating the interferometer beam spot positions with respect to center on the ETM surfaces. For the LHO X-end ETM, the interferometer beam was measured to be 14.3 mm below the center and 16.2 mm to the left of the center.

Using (11) and  $0.94 \times 10^{-4}/\text{mm}^2$  for  $M/I$ , the expected relative change (after relative to before) in the X/Y comparison factor is  $-16.8 \times 10^{-4}$  (-16.8 hop) for the vertical Pcal spot position move and  $-19.0 \times 10^{-4}$  (-19.0 hop) for the horizontal move.

Sign convention for spot position: up (+Vert on SUS screens) is positive for pitch and farther to the left (+Trans on SUS screens) is positive for yaw.

Current A2L values



a2l\_lookup('PIT',4.0)

Spot is -14.3 mm from the PIT center of the optic

a2l\_lookup('PIT',4.60)

Spot is -17.1 mm from the PIT center of the optic

a2l\_lookup('YAW',4.4)

Spot is 16.2 mm from the YAW center of the optic

a2l\_lookup('YAW',3.2)

Spot is 11.8 mm from the YAW center of the optic

**Figure 1.** Excerpt from email from J. Driggers regarding LHO interferometer beam positions on the test mass surfaces based on angle-to-length measurements.