Draft version April 30, 2026
Typeset using LaTeX **default** style in AASTeX7.0.1

Microlensing Fourier Convention Inconsistency

Mick Wright[1,2] and Justin Janquart[3,4]

[1] *Department of Physics, Institute for Gravitational and Subatomic Physics (GRASP), Utrecht University, Princetonplein 1, NL-3584 CC Utrecht, The Netherlands*

[2] *Nikhef—National Institute for Subatomic Physics, Science Park, NL-1098 XG Amsterdam, The Netherlands*

[3] *Center for Cosmology, Particle Physics and Phenomenology - CP3, Université Catholique de Louvain, Louvain-La-Neuve, B-1348, Belgium*

[4] *Royal Observatory of Belgium, Avenue Circulaire, 3, 1180 Uccle, Belgium*

ABSTRACT

The following outlines an inconsistency in Fourier convention that impacts isolated point mass analyses undertaken prior to GWTC-4.0. This inconsistency arises due to the differing conventions used in the implementations of the amplification factor and the waveform model and antenna pattern. This may be corrected by complex conjugation of the amplification factor before it is applied. However, this has a significant impact on the resultant waveform and consequently parameter estimation recoveries are significantly altered when the inconsistency is fixed. We detail this issue and the correction in this work.

1. INTRODUCTION

As part of the suite of analyses ran on gravitational wave signals since GWTC-2.1, the LIGO–Virgo–KAGRA (LVK) Collaborations have undertaken parameter estimation studies on each signal under the hypothesis of lensing by an isolated point mass as a proxy for any wave-optics effects caused by lensing objects. In this work, we detail an inconsistency in Fourier transform implementation that was uncovered during the process of putting together the results for GWTC-4.0 which has impact on those runs performed before said catalog.

2. DEFINITIONS

2.1. *Fourier Transform Conventions*

There are a number of conventions for the construction of the Fourier transform which differ in sign and pre-factor. Two commonly used conventions, which are of relevance to this work, are the so-called *'Physics'* and *'Engineering'* conventions.

Convention. *Physics*
The Fourier transform is defined as

$$\tilde{x}(f) = \int x(t) e^{2\pi i f t} \mathrm{d}t, \tag{1}$$

for a function x, where f and t denote frequency and time respectively. The inverse Fourier transform is consequently defined as

$$x(t) = \int \tilde{x}(f) e^{-2\pi i f t} \mathrm{d}f. \tag{2}$$

Convention. *Engineering*
The Fourier transform is defined as

$$\tilde{x}(f) = \int x(t) e^{-2\pi i f t} \mathrm{d}t, \tag{3}$$

where the symbols have retained their meanings from the previous. The inverse Fourier transform is consequently defined as

$$x(t) = \int \tilde{x}(f) e^{2\pi i f t} \mathrm{d}f. \tag{4}$$

As can be seen, these conventions differ only in the sign of the exponential.

2.2. *Amplification Factor*

The effects of gravitational lensing on a gravitational wave signal are encoded in the amplification factor, $\tilde{F}(f)$, which is defined as the ratio of the wave amplitudes of the resulting lensed signal and the original source signal. The form of this function is dependent upon the geometries of both the system and the lens itself, however, the relationship between the observed lensed strain, $\tilde{h}_L(f)$, and the emitted unlensed strain, $\tilde{h}(f)$, is simply

$$\tilde{h}_L(f) = \tilde{F}(f) \times \tilde{h}(f). \tag{5}$$

3. CONVENTION DISCREPANCY

Papers describing gravitational lensing—such as Takahashi & Nakamura (2003) or Ezquiaga et al. (2021)—typically give the amplification factor, in terms of the reesults of the diffraction integral, as

$$\tilde{F}(w, \vec{y}) = \frac{w}{2\pi i} \int \exp\left(iwT(\vec{x}, \vec{y})\right) \mathrm{d}^2\vec{x}, \tag{6}$$

where $\vec{x}$ and $\vec{y}$ are dimensionless forms of the image and source positions respectively, w denotes a dimensionless form of the frequency, and the function T describes the dimensionless time delay resulting from the lensing which arises as a combination of the geometrical path difference after deflection as well as the Shapiro delay resulting from passing close by the lensing object.

In cases where the wavelength of the signal is similar to the Schwarzschild radius of the lensing object, this expression must be fully evaluated as the diffraction effects are present—this is termed wave optics—and manifests as a distorted signal rather than multiple resolvable signals. Conversely, outside of this regime, the geometric optics approximation may be applied. In this approximation, the only points assumed to contribute to the above integral are the stationary points of the time delay function. This reduces the integration to a discrete summation over the images produced by the lensing, i.e.

$$F(w, \vec{y}) = \sum_j |\mu_j|^{1/2} \exp\left(iwt_j - i\pi n_j\right), \tag{7}$$

where μ_j, t_j, and n_j are respectively the magnification, time delay, and so-called Morse index of the j^{th} image—the lattermost term reflecting the image type which is determined by the nature of the corresponding stationary point of the time delay function. The time delays in this approximation may be smaller than the duration of the signal in which case the resulting signal appears to have beating patterns due to the interference of the overlapping signals or may be larger resulting in wholly resolvable gravitational wave signals for each image.

However, as may be seen from above, these expressions use the Physics convention for the Fourier transform. Implementations of the waveforms and the antenna patterns of the detectors, in LAL (LIGO Scientific Collaboration et al. 2018) and BILBY (Ashton et al. 2019) respectively, use the Engineering Fourier convention, meaning that this should be corrected in implementation of the amplification factor. However, prior to the GWTC-4.0 lensing analyses, this correction was not implemented due to the convention difference going unnoticed.

The correction is relatively simple to implement. Correcting the convention and re-adjusting the normalisation prefactor to continue to ensure that the amplification tends to unity as the frequency tends to zero yields

$$F(w, \vec{y}) = -\frac{w}{2\pi i} \int \exp\left(-iwT(\vec{X}, \vec{y})\right) d^2\vec{x} \tag{8}$$

i.e. the complex conjugate of Equation 6. Illustrations of this are shown in Figure 1.

4. IMPACTS ON PRIOR ANALYSES

In the context of analyses that consider temporally resolvable signals, the impact is minimal, it is equivalent to a reversal of the polarisation angle. However, the impact on those analyses that consider a single signal or multiple signals that are less than the duration of the signal such that the signal is analysed in totality at a single time, the effects of the change are more severe. This is illustrated in Figure 2.

The considerable difference in waveform morphology under the two conventions imparts considerable differences in parameter estimation recoveries of signals. An example is shown in Figure 3 for the event GW200208_130117 (Abbott et al. 2023). During the GWTC-3 based lensing analyses which were impacted by the convention inconsistency, this

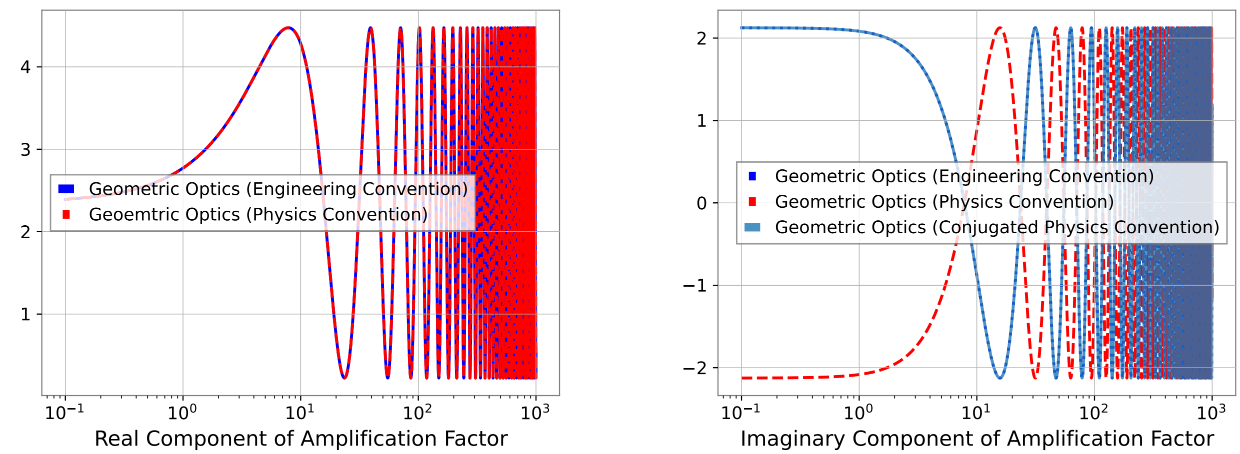

Figure 1. Comparisons of computing the amplification factor (for the isolated point mass model) in the engineering and physics conventions in the geometric optics case. The left panel shows the real component and the right the imaginary, with the latter panel also showing the conjugation demonstrating that conjugation of the Physics convention yields the Engineering convention.

event was found to be the candidate with the most support during the run (Collaboration et al. 2023). Follow-up analyses (Janquart et al. 2023), however, ultimately concluded that the apparent support was likely the result of transient noise in the Hanford detector. However, re-analysis using consistent conventions yielded significantly weaker constraints on lensing parameters and a subsequent reduction in Bayes factor support. Whilst this does not alter the ultimate conclusion of prior analyses, it does alter the reasoning. Furthermore, it does open the possibility that the reverse may also occur, i.e. an event that previously had no apparent support may demonstrate support with the corrected convention.

REFERENCES

Abbott, R., Abbott, T., Acernese, F., et al. 2023, Physical Review X, 13, doi: 10.1103/physrevx.13.041039

Ashton, G., Hübner, M., Lasky, P. D., et al. 2019, ApJS, 241, 27, doi: 10.3847/1538-4365/ab06fc

Collaboration, T. L. S., the Virgo Collaboration, the KAGRA Collaboration, et al. 2023, Search for gravitational-lensing signatures in the full third observing run of the LIGO-Virgo network. https://arxiv.org/abs/2304.08393

Ezquiaga, J. M., Holz, D. E., Hu, W., Lagos, M., & Wald, R. M. 2021, Phys. Rev. D, 103, 064047, doi: 10.1103/PhysRevD.103.064047

Janquart, J., Wright, M., Goyal, S., et al. 2023, MNRAS, 526, 3832, doi: 10.1093/mnras/stad2909

LIGO Scientific Collaboration, Virgo Collaboration, & KAGRA Collaboration. 2018, LVK Algorithm Library - LALSuite, Free software (GPL), doi: 10.7935/GT1W-FZ16

Takahashi, R., & Nakamura, T. 2003, Astrophys. J., 595, 1039, doi: 10.1086/377430

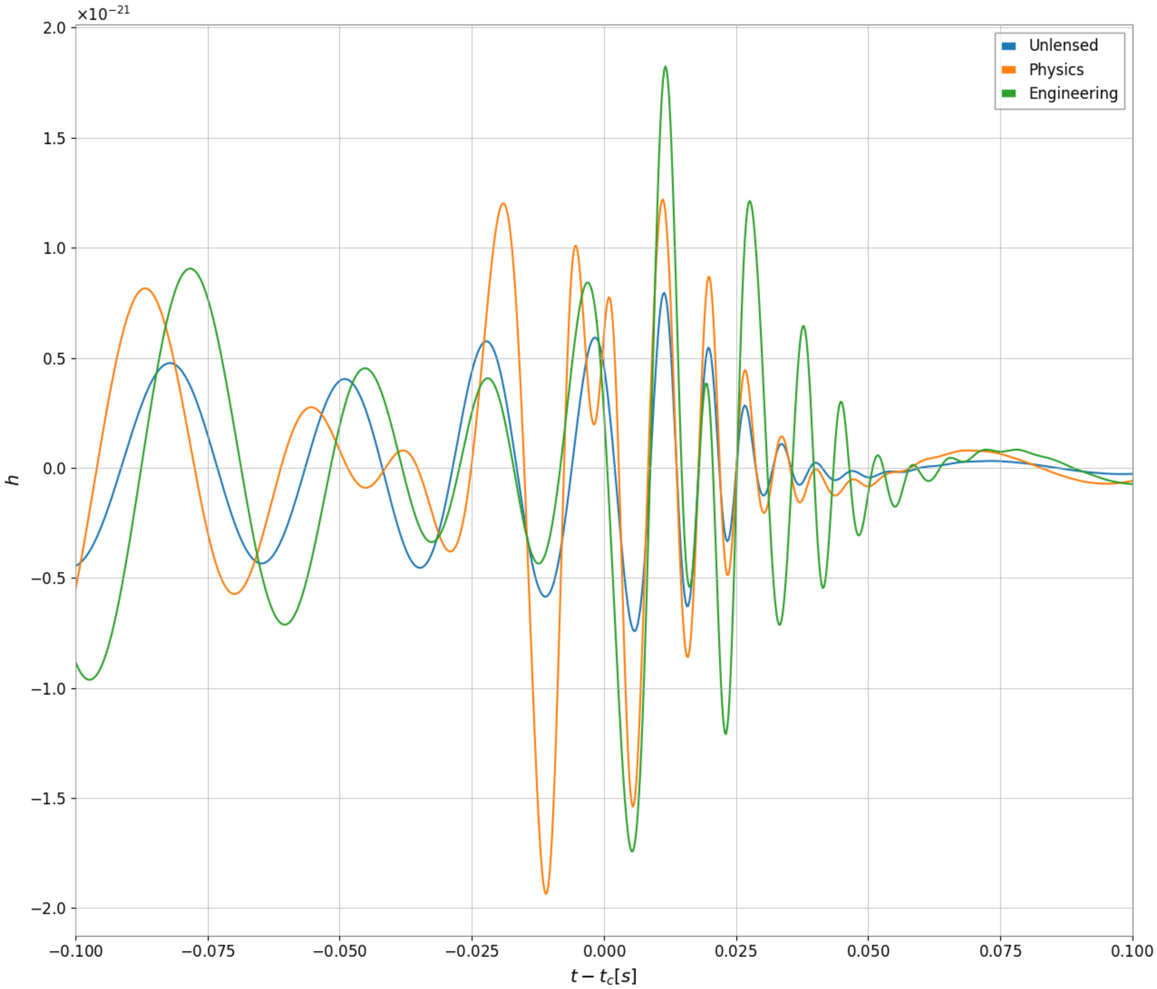

Figure 2. Illustration of the resultant lensed waveforms using the two different Fourier conventions for the amplification factor. The original unlensed waveform is also shown for comparison. As can be seen there are significant differences between the two lensed waveforms which yields similar levels of difference in parameter estimation based investigations.

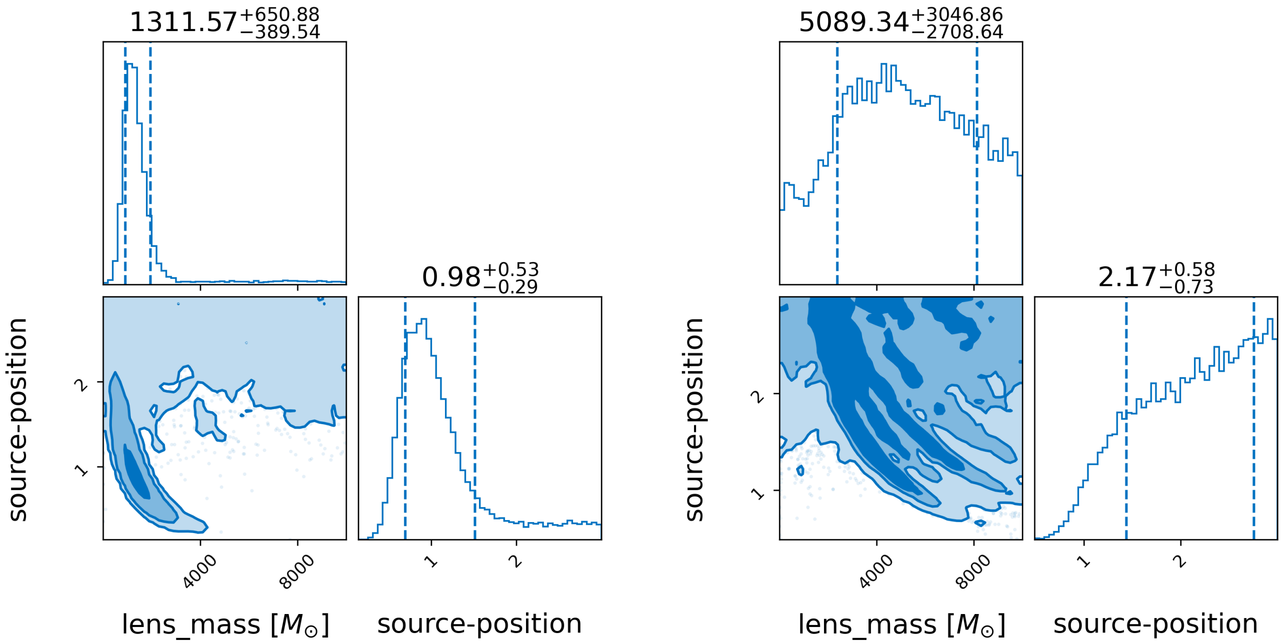

Figure 3. Comparison on the results of parameter estimation under the isolated point mass hypothesis with the incorrect Physics convention (left) and the correct Engineering convention (right) for the amplification factor implementation. As may be seen, the corrected convention yields significantly less of a constraint on the lensing parameters.