Modeling Compact Mergers detected by LIGO with Bivariate B-splines

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Hierarchical Bayesian Modeling

$$p(\vec{\Lambda}|\{\vec{d}_i\}) = \pi(\vec{\Lambda}) \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} p_{\rm pop}({}^j\vec{\theta}_i|\vec{\Lambda})\frac{1}{\pi(\vec{\theta})}}{\int d\vec{\theta}\, p_{\rm det}(\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\Lambda})}$$

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Modeling compact mergers

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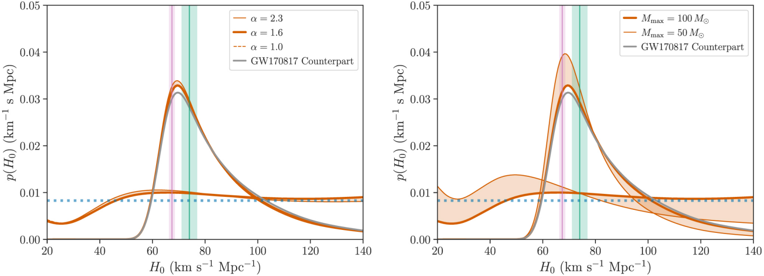

Figure: B.P. Abbott et al. 2021

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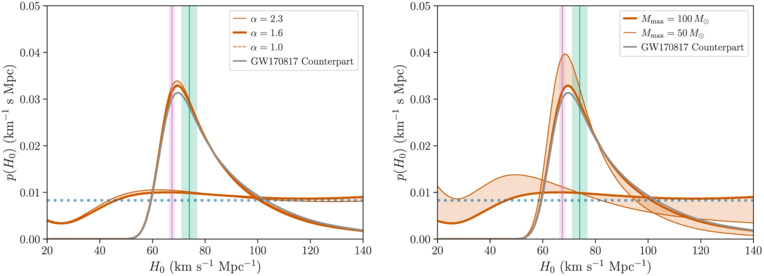

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- ...all of these rely on population models

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  - *pros*: data-driven, introduces less bias, reveal substructures

Population models

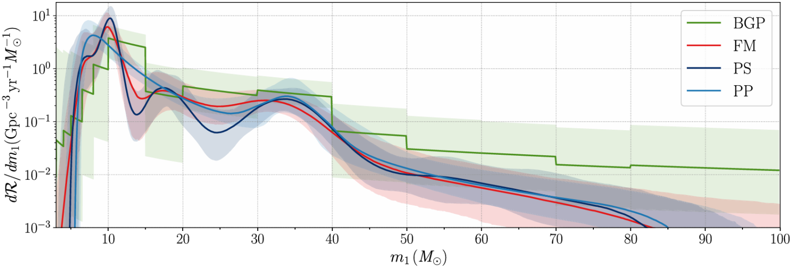

Figure: R. Abbott et al. 2023

Figure: *Left*: parametric models. *Right*: non-parametric models.

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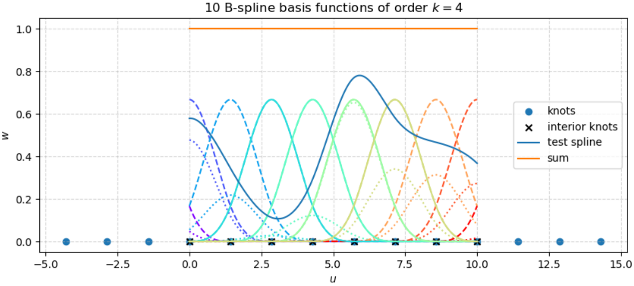

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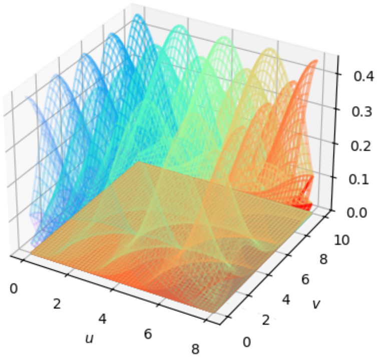

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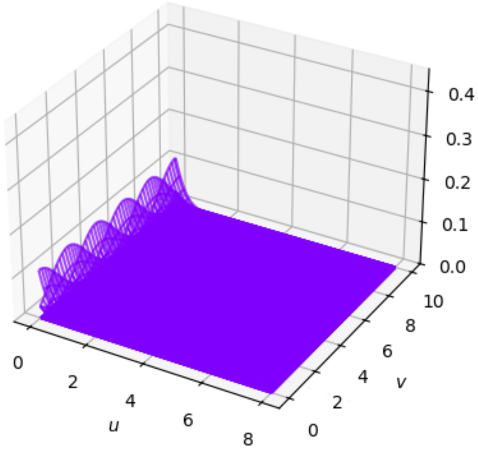

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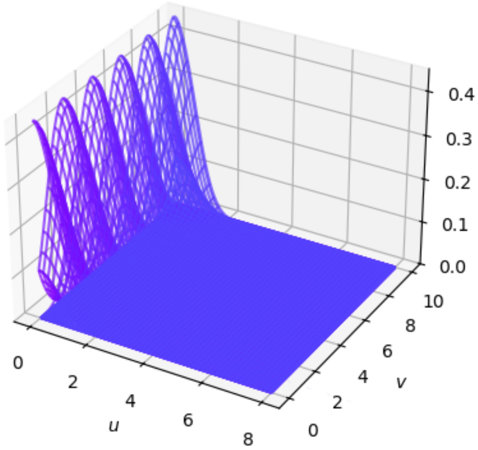

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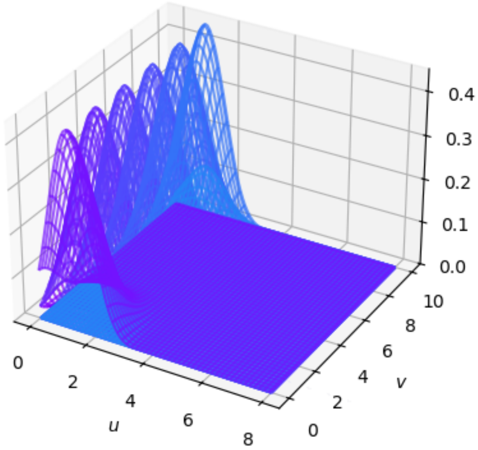

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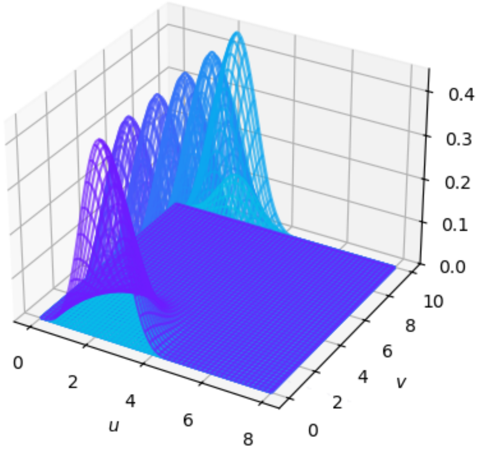

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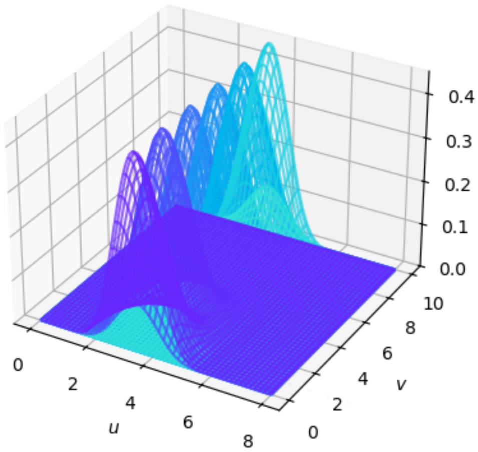

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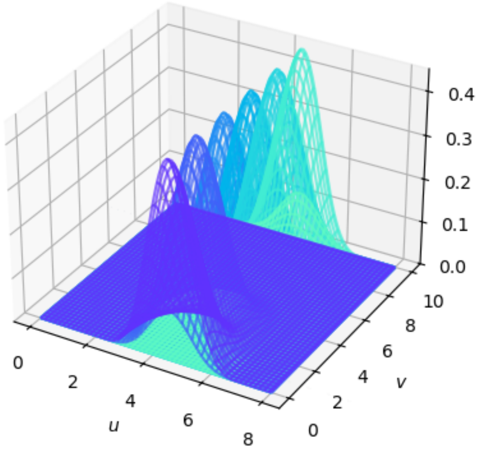

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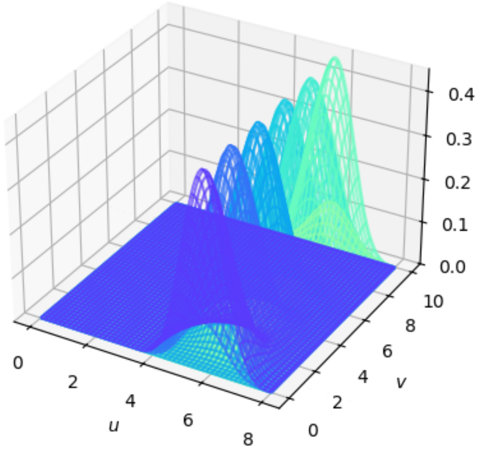

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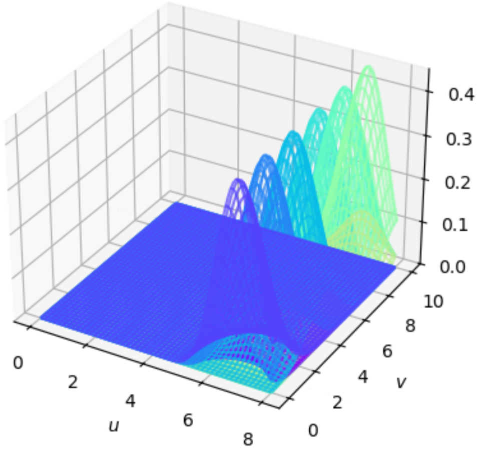

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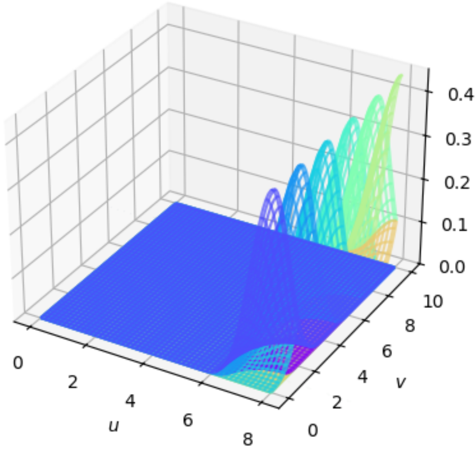

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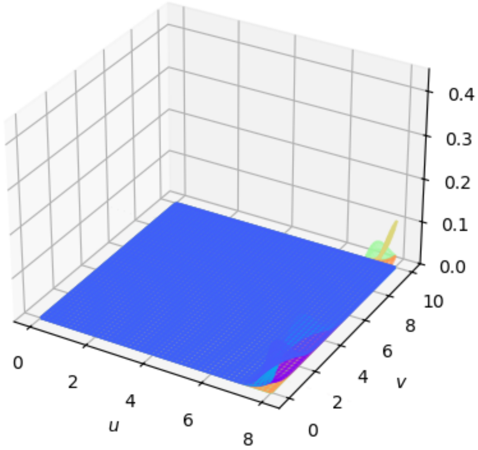

Bivariate B-spline models

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\begin{aligned}
p(a_1, a_2, \cos\theta_1, \cos\theta_2|\alpha) &= \sum_{i,j}^{I,J} \alpha_{ij} B_{i,k}(a_1) B_{j,k}(\cos\theta_1) \\
&\times \sum_{\ell,m}^{L,M} \alpha_{\ell m} B_{\ell,k}(a_2) B_{m,k}(\cos\theta_2) \\
&= \mathcal{B}_1(a_1, \cos\theta_1|\alpha) \mathcal{B}_2(a_2, \cos\theta_2|\alpha).
\end{aligned}
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$$\chi_{\text{eff}} = \frac{(m_1\vec{a}_1 + m_2\vec{a}_2)\cdot\hat{L}}{m_1+m_2} = \frac{m_1 a_1 \cos\theta_1 + m_2 a_2 \cos\theta_2}{m_1+m_2} = \frac{a_1\cos\theta_1 + q a_2 \cos\theta_2}{1+q},$$

$\hat{L}$ is a unit vector parallel to the orbital angular momentum of the binary system

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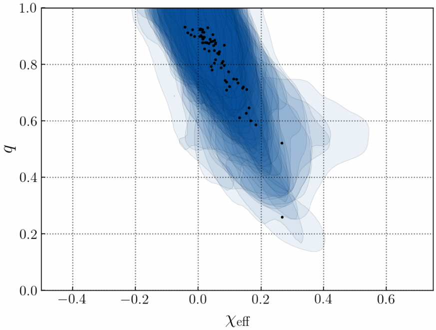

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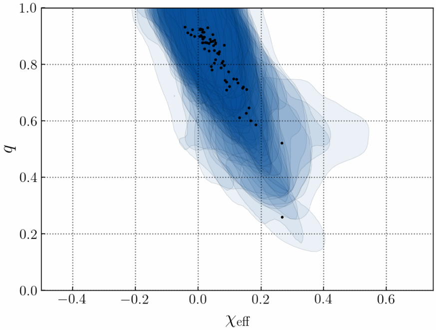

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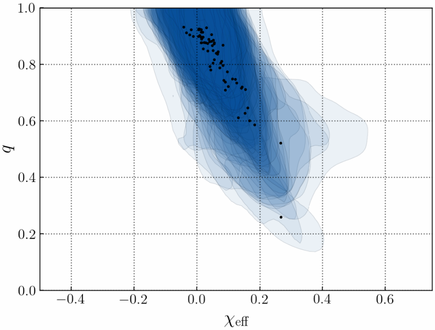

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- Break mass degeneracy between isolated and dynamical formation → model as bivariate spline!
- Inform model building, disentangle formation channels

Backup

Defining B-splines

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and for $k > 1$,

$$B_{i,k} = \omega_{i,k} B_{i,k-1} + (1 - \omega_{i+1,k}) B_{i+1,k-1},$$

where

$$\omega_{i,k}(x) \equiv \frac{x - u_i}{u_{i+k-1} - u_i}$$

Current work

- Choosing priors

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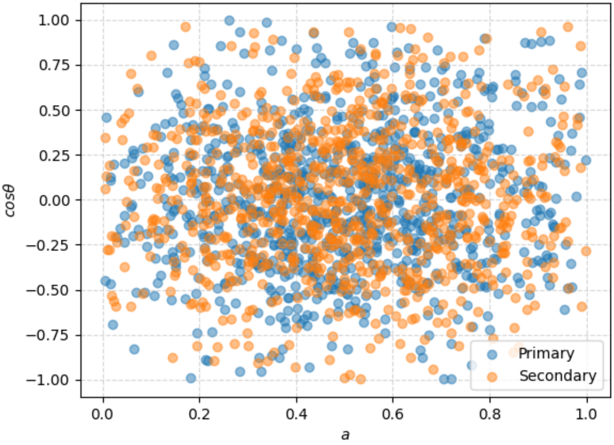

Choosing priors

Model	Degrees of freedom	Prior
1	100	$\mathcal{N}(0,1)$
2	100	$\mathcal{N}(\alpha_s,1)$
3	196	$\mathcal{N}(0,1)$
4	196	$\mathcal{N}(\alpha_s,1)$

Table: Choices for priors on the coefficients α. When a normal prior is centered on α_s, this means each α_{ij} is centered on a value α_s^{ij}. The values α_s are found by minimizing the least squares objective $S = \|y - B\alpha\|^2$, where B is the basis matrix containing the basis functions of the B-spline, and y contains the true values of the distribution.

Bivariate spline fit