Evaluating the Co-Precessing Frame Approximation for Eccentric Binaries

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Background

What are eccentricity and precession and how do they affect waveforms?

Eccentricity

Eccentricity (e): number varying from 0-1 for bound orbits

- Tendency to circularize
- Normally assume quasi-circularity in waveform detection and analysis
- With increasing detector sensitivity in low frequency range, we can relax this assumption

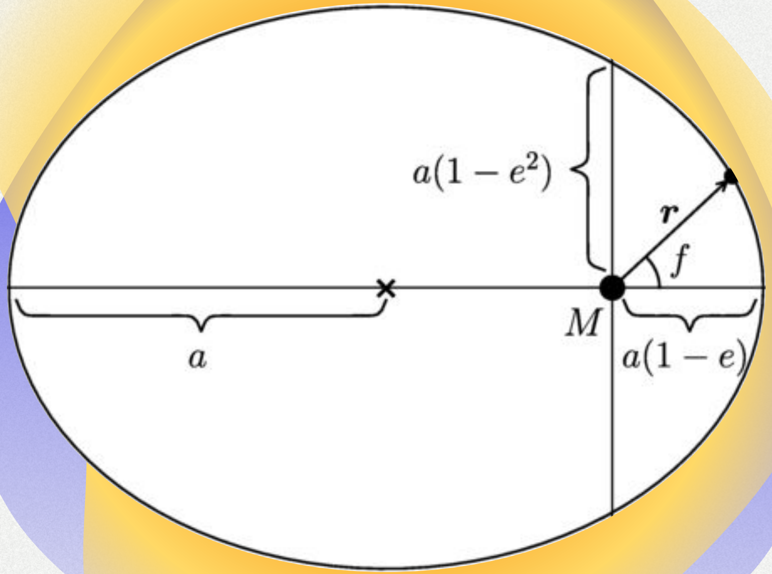
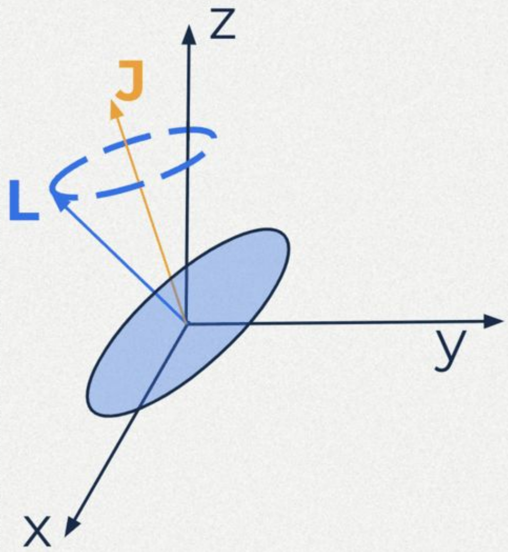


Fig 1, Fumagalli et al. 2023

Precession



- Occurs when black hole spins are misaligned with **L**
- **L** and spins precess around total angular momentum vector **J**
- Orbital plane is like a spinning top
- Complicates GW by modulating phase and amplitude

L: orbital angular momentum

J: total angular momentum

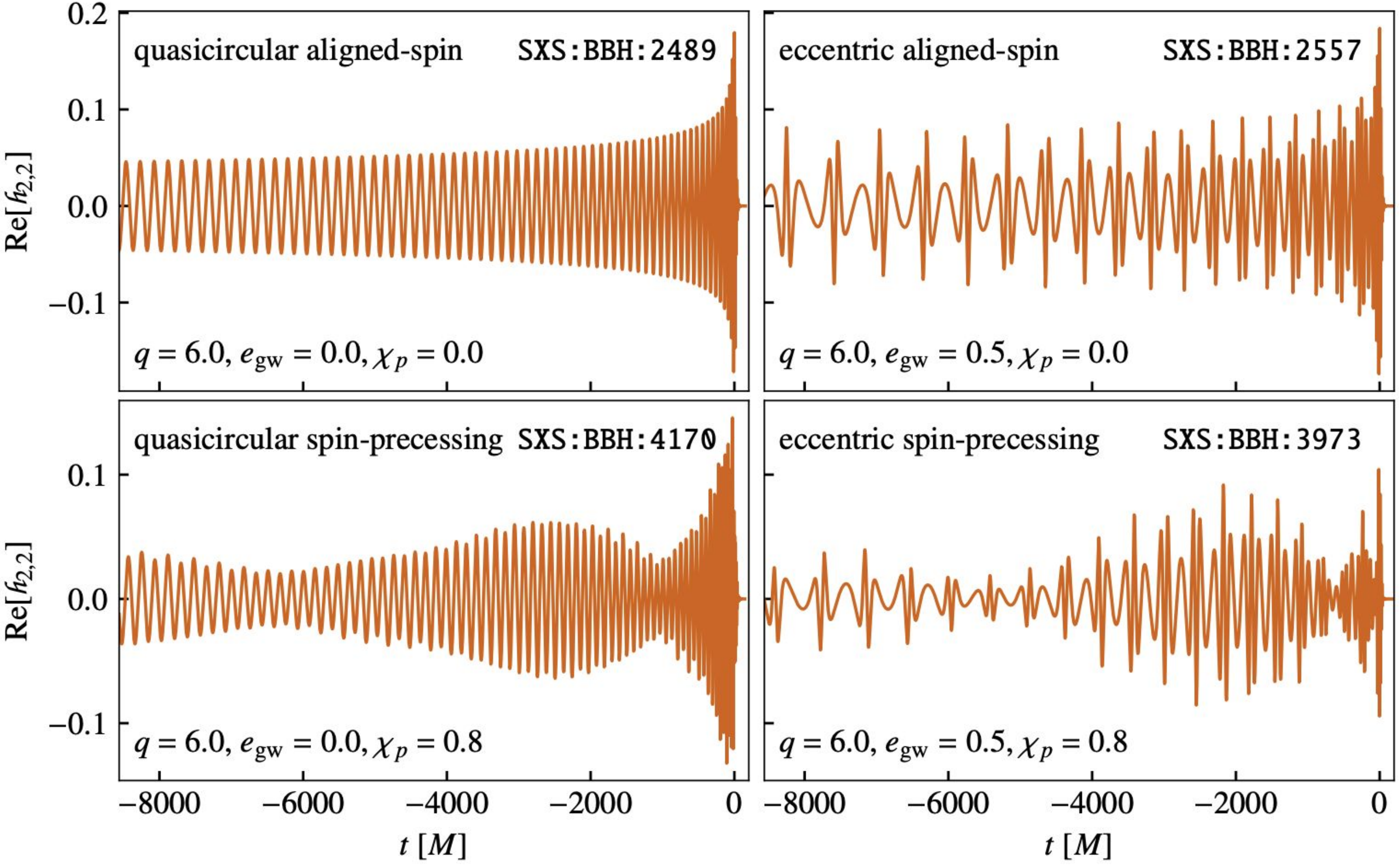
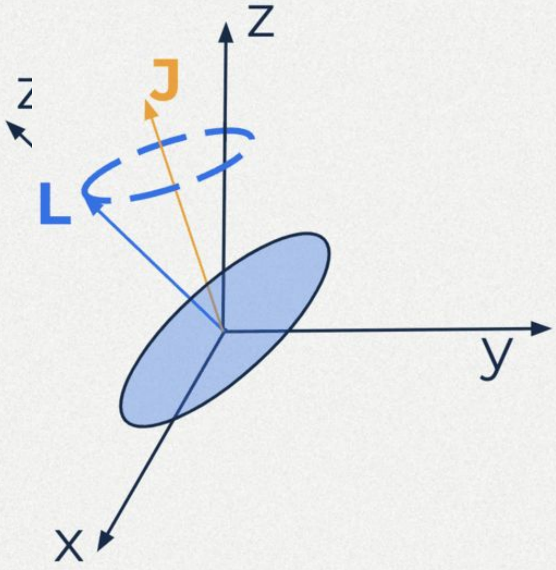


Figure 1, Shaikh et al. 2025

The Co-Precessing Frame



L: orbital angular momentum

J: total angular momentum

- Possible to construct a reference frame that simplifies precessing waveforms
- This new frame travels with precession → **co-precessing frame**
  - constructed so **z** is always aligned with **L**
- Shift from inertial to co-precessing frame = "untwisting"
- Use code package scri* to untwist
- Approximated to remove precession-related amplitude modulation for QC systems, but not well tested for eccentric

*Boyle et al, DOI: 10.5281/zenodo.4041972

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Goals

What question are we trying to answer?

Research Question

Is the co-precessing frame approximation accurate for eccentric binaries?

Or, if not, can we characterize its inaccuracies?

Note: we are particularly studying the inspiral

Two Motives:

1) **Modelling:**
    - If approximation holds, we can reverse transformation for easier modelling of eccentric precessing systems
2) **Formation:**
    - Eccentric and precessing binaries likely formed through dynamical capture!

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Methods

How did we test the co-precessing frame approximation?

Our Process

: 3 types of comparisons using NR sims (184 ecc, 27 eccprec)

1

NR Non-prec Inertial vs. Copr

Compare co-precessing and inertial frame non-precessing GWs (QC and eccentric)

2

NR Ecc-prec vs. Ecc

Compare untwisted eccentric NR waveforms to non-precessing NR with similar parameters

3

NR vs. SEOBNRv5EHM

Generate non-precessing matches for eccentric precessing NR sims with SEOBNRv5EHM (aka v5EHM) and compare parameters

What does “compare” mean quantitatively?

Quantitative Comparison: Mismatches

Mismatch (μ): real number between 0 and 1

- Quantitative measure of waveform similarity
  - 0: same waveform
  - 1: fully out of phase
- Normalized inner product between waveforms

$$<h_1, h_2> = 4\Re \int_{f_{min}}^{f_{max}} \tilde{h_1}(f)\tilde{h_2^*}(f)\, df$$

$$\mu = 1 - \max_{t_c, \phi_0} \frac{<h_1, h_2>}{\sqrt{<h_1, h_1><h_2, h_2>}}$$

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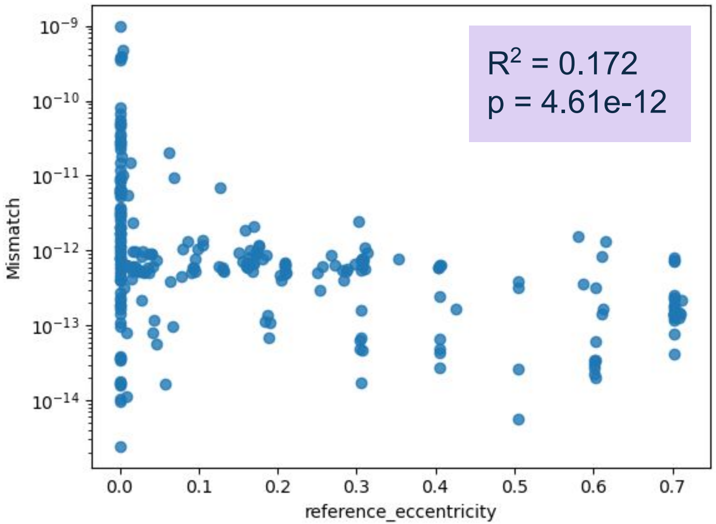
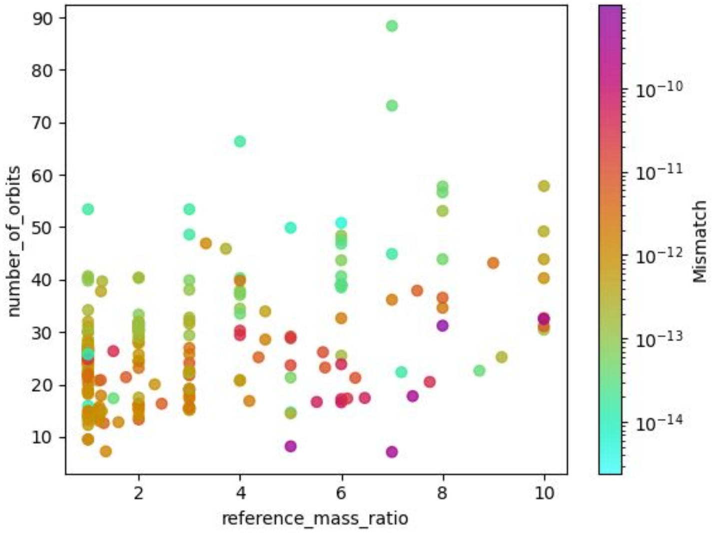
Results

What we found!





NR Non-Precessing Comparisons

Negligible mismatches! *

By definition, co-precessing frame = inertial frame for non-precessing systems — waveforms should be identical ($\mu \sim 0$)

No correlation between eccentricity and mismatch

- validates scri's ability to find co-precessing frame regardless of eccentricity
- relationship with mass ratio/number of orbits suggest variation in μ is a result of NR simulation resolution/noise (plus random variation)

However, this analysis does not account for effects unique to precessing systems (e.g. ecc/prec interplay)

For that, we need to **analyze precessing simulations**

***one outlier removed from plots**

NR Matches

- 27 undeprecated eccentric & precessing NR waveforms

<u>Goal:</u> find non-precessing NR sims (matches) with similar parameters to untwisted eccentric precessing waveforms

- Matching criteria defined in two ways →
- Despite loose parameters...

only three matches!

Criteria 1

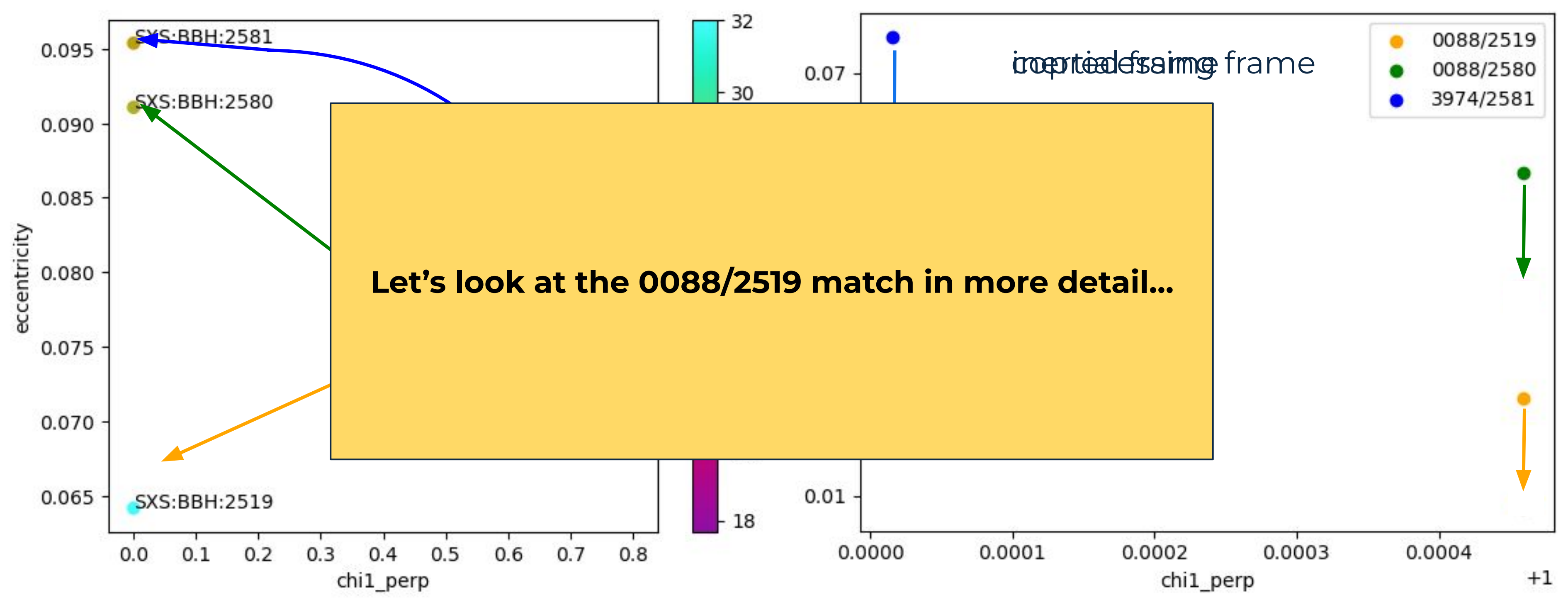
$$|\chi_{eff_{ep}} - \chi_{eff_{match}}| < 0.1$$
$$|q_{ep} - q_{match}| < 0.4$$
$$|\Omega_{ep} - \Omega_{match}| < 0.0005$$
$$0.7 < \frac{e_{ep}}{e_{match}} < 1.3$$
$$|\chi_{1_{\perp match}} + \chi_{2_{\perp match}}| < 0.01$$

Criteria 2

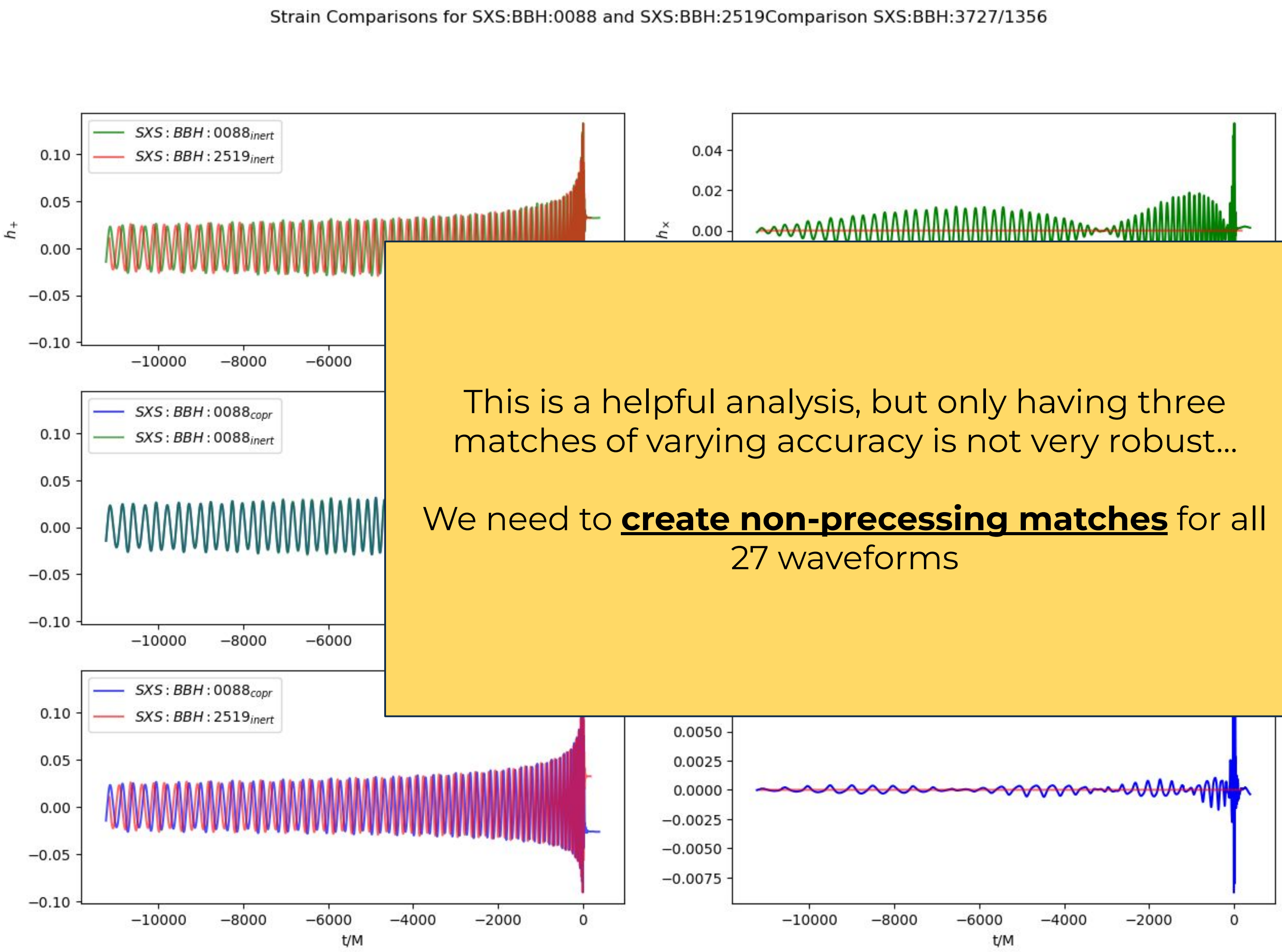
$$|N_{ep} - N_{match}| < 15$$
$$|\ell_{ep} - \ell_{match}| < \frac{\pi}{8}$$
$$0.6 < \frac{e_{ep}}{e_{match}} < 1.4$$

ep = eccentric precessing

NR Matches



Let's look at the 0088/2519 match in more detail...



This is a helpful analysis, but only having three matches of varying accuracy is not very robust...

We need to **create non-precessing matches** for all 27 waveforms

Strain Comparison

Green: 0088 inert

Blue: 0088 copr

Red: non-prec match (inert)

- Much more similar (~zero) in co-precessing frame
- But wiggles remain — lingering precession effects
- these wiggles follow inertial frame amplitude envelope

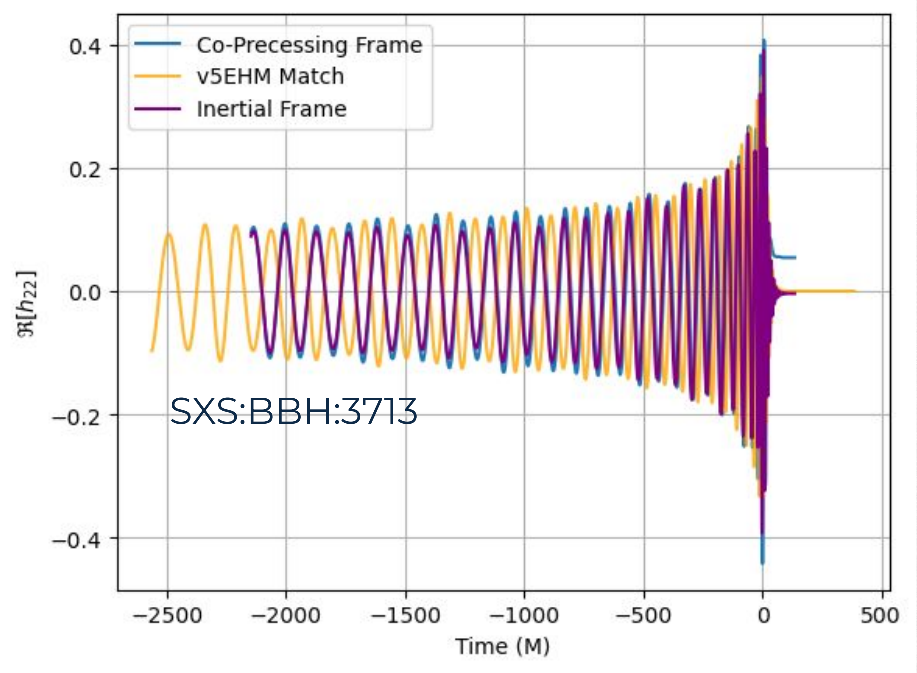
v5EHM Comparisons

We can use matches SEOBNRv5EHM model to generate matches for all 27 GWs!

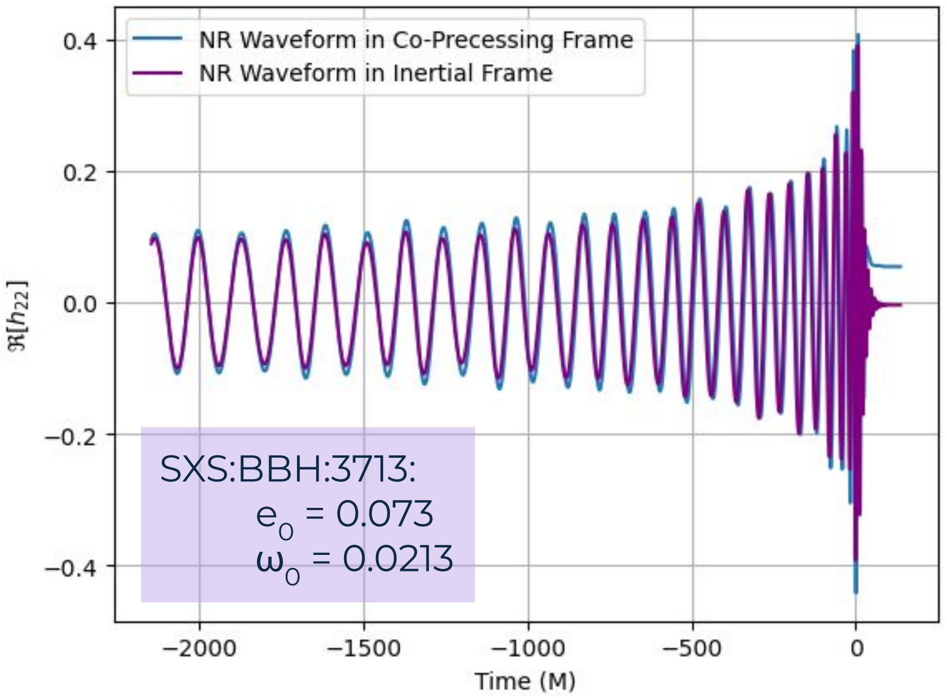
Feeding v5EHM NR params does not work well →

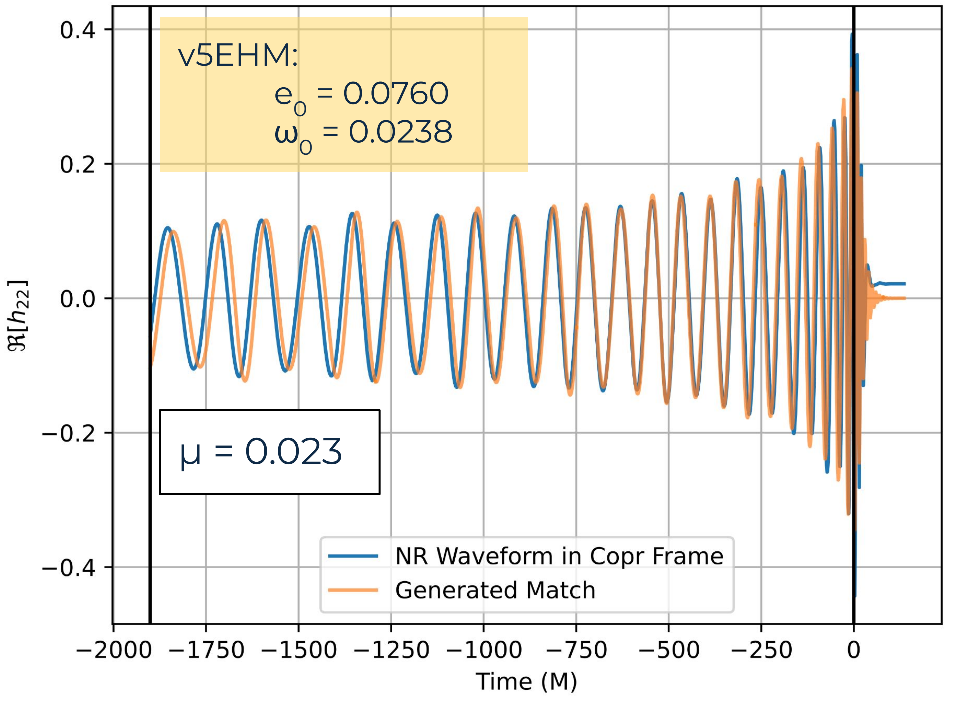
Instead, we fit over the parameters and compare resulting eccentricities:

Minimize time domain "mismatch" between untwisted eccprec NR sim and a generated v5EHM waveform and over **starting orbital frequency** and **eccentricity**



v5EHM Comparisons





Process is working fairly well, but in early stages — only run on a few waveforms

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Conclusions + Future Work

Our initial conclusions and next steps

Conclusions

1) From non-precessing mismatch analysis:

Scri can correctly find co-precessing frame regardless of effects from eccentricity in isolation

2) From NR eccprec vs. NR ecc match comparisons:

Untwisted eccentric waveforms are much more similar to non-precessing counterparts, but some precession effects remain

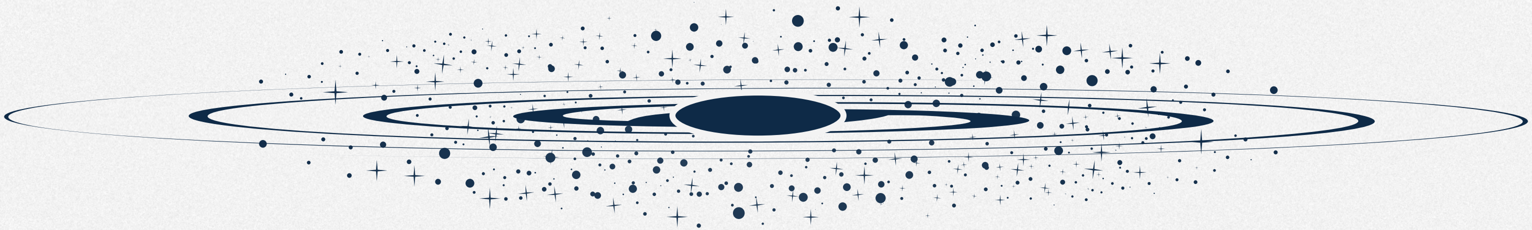
3) From NR eccprec vs. v5EHM match:

Untwisting may affect eccentricity of waveform

→ needs further investigation

Up Next?

- Finish generating v5EHM matches
- Compare eccentricity between NR precessing waveforms and non-precessing v5EHM matches
- Compare dynamics for 27 eccentric and precessing NR simulations between inertial and co-precessing frames



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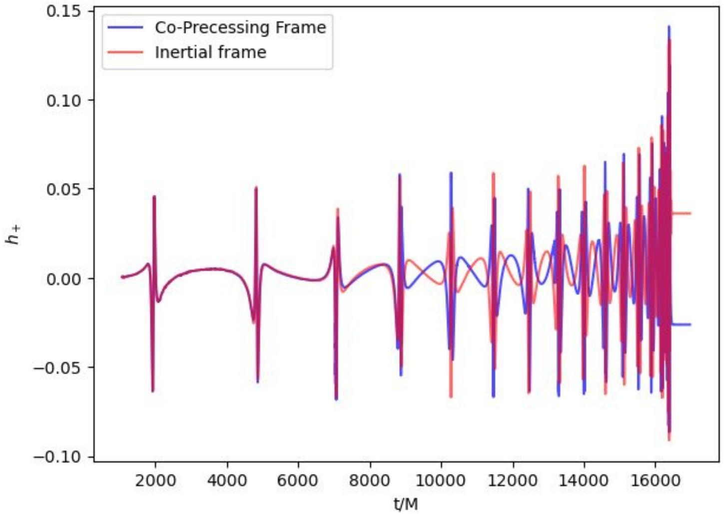
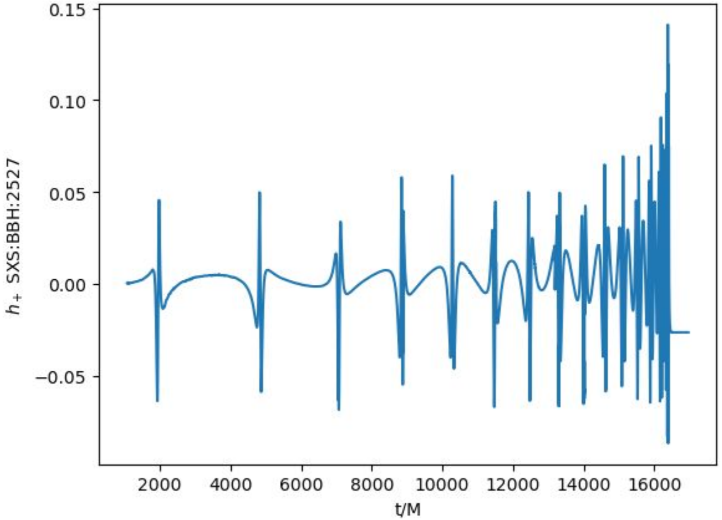
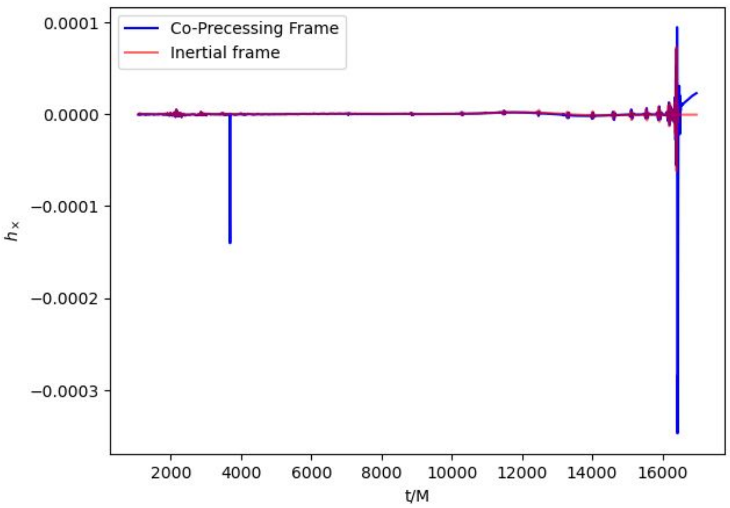
Questions?

Weird simulation we found while doing mismatches!








```
reference_eccentricity                0.798873
reference_mass_ratio                       1.0
reference_mean_anomaly                3.775349
number_of_orbits                     19.918326
reference_orbital_frequency_mag       0.000453
Name: SXS:BBH:2527, dtype: object
```

~0.94 mismatch

Something is clearly wrong with the copr frame transformation

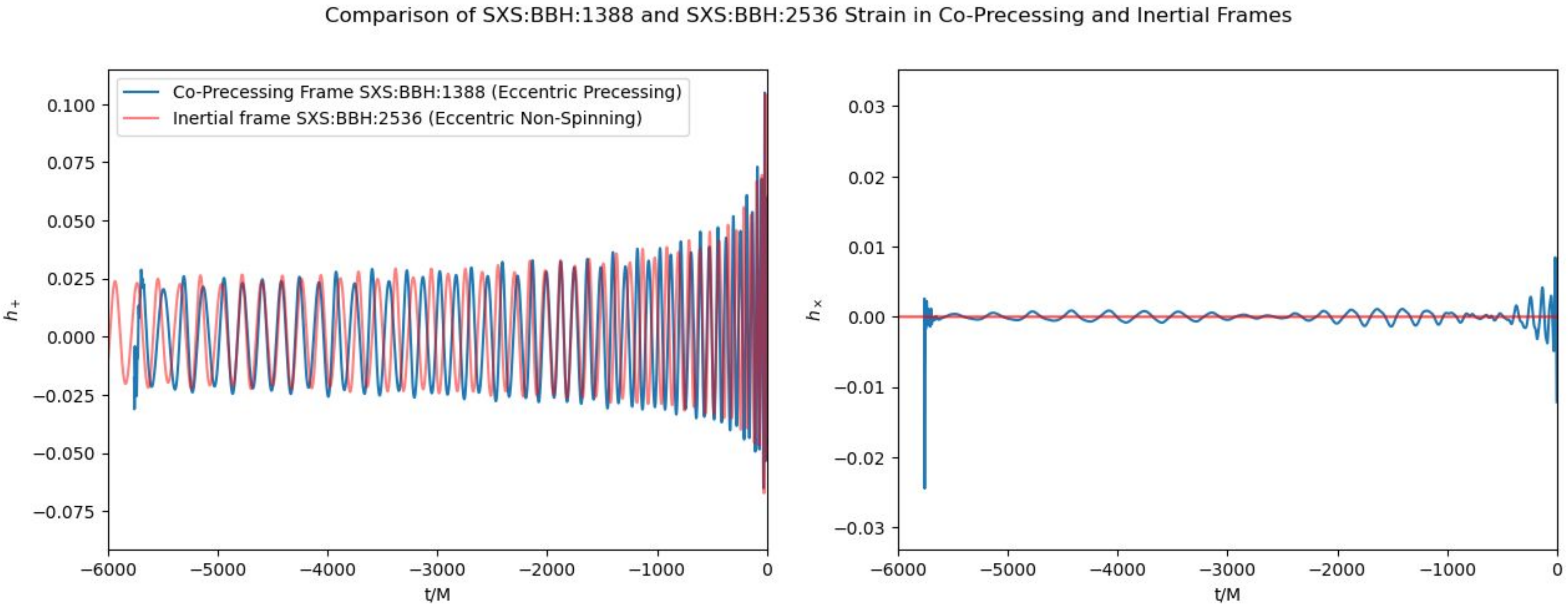
Zoom whirl?

NR Matches

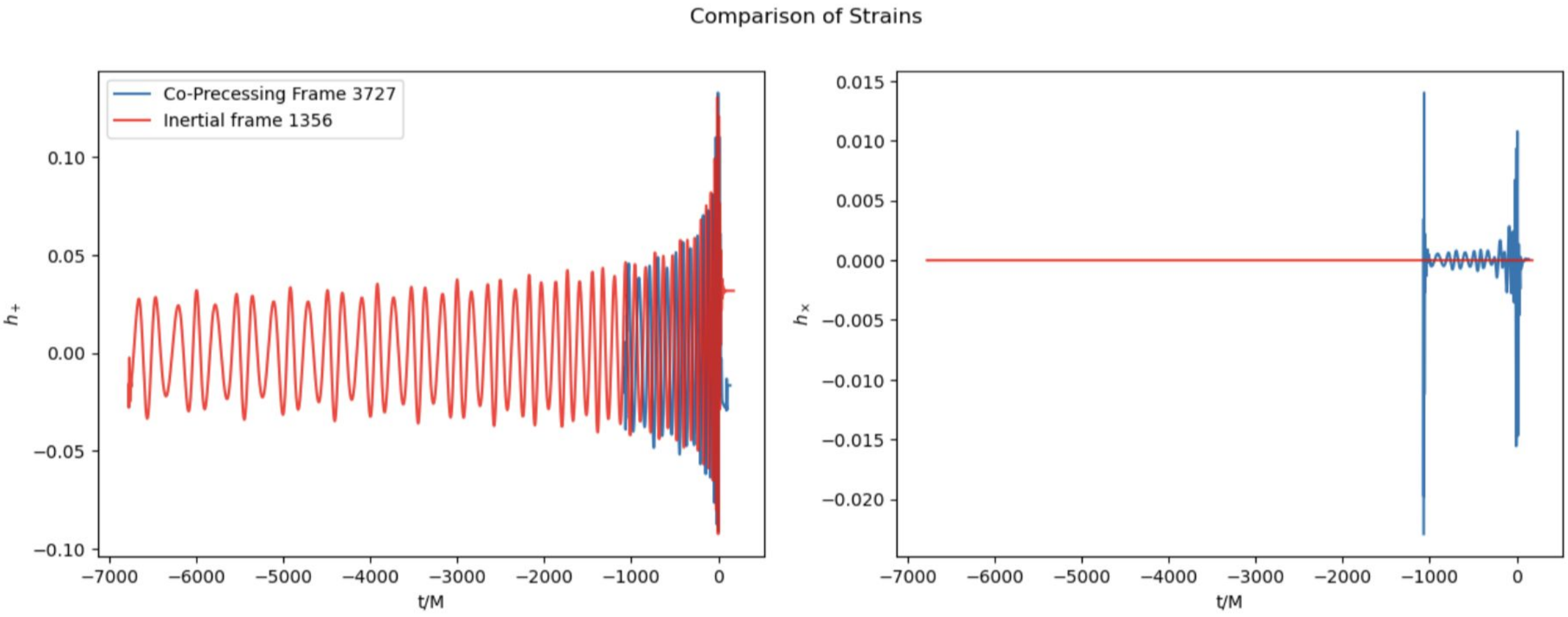
Eccprec simulation	Ecc non-prec match	Inertial Frame μ	Copr/Inert μ
SXS:BBH:0088	SXS:BBH:2519	0.0239	0.00837
SXS:BBH:0088	SXS:BBH:2580	0.0559	0.0388
SXS:BBH:3974	SXS:BBH:2581	0.0752	0.0324

	eccentricity	chi1_perp	chi2_perp	chi_eff	mean anomaly	# of orbits
SXS:BBH:0088	0.074224	4.995056e-01	8.566428e-08	-1.014844e-04	1.408540	31.874570
SXS:BBH:2519	0.064158	9.788037e-11	4.810103e-11	-4.438172e-08	1.361416	32.005703
SXS:BBH:2580	0.091067	5.205124e-10	4.534452e-10	-2.518567e-07	0.138174	26.833510
SXS:BBH:3974	0.069397	7.999421e-01	2.746283e-06	-2.386863e-04	2.204960	17.691507
SXS:BBH:2581	0.095376	2.498750e-08	1.381648e-09	1.298104e-06	2.509668	26.066627

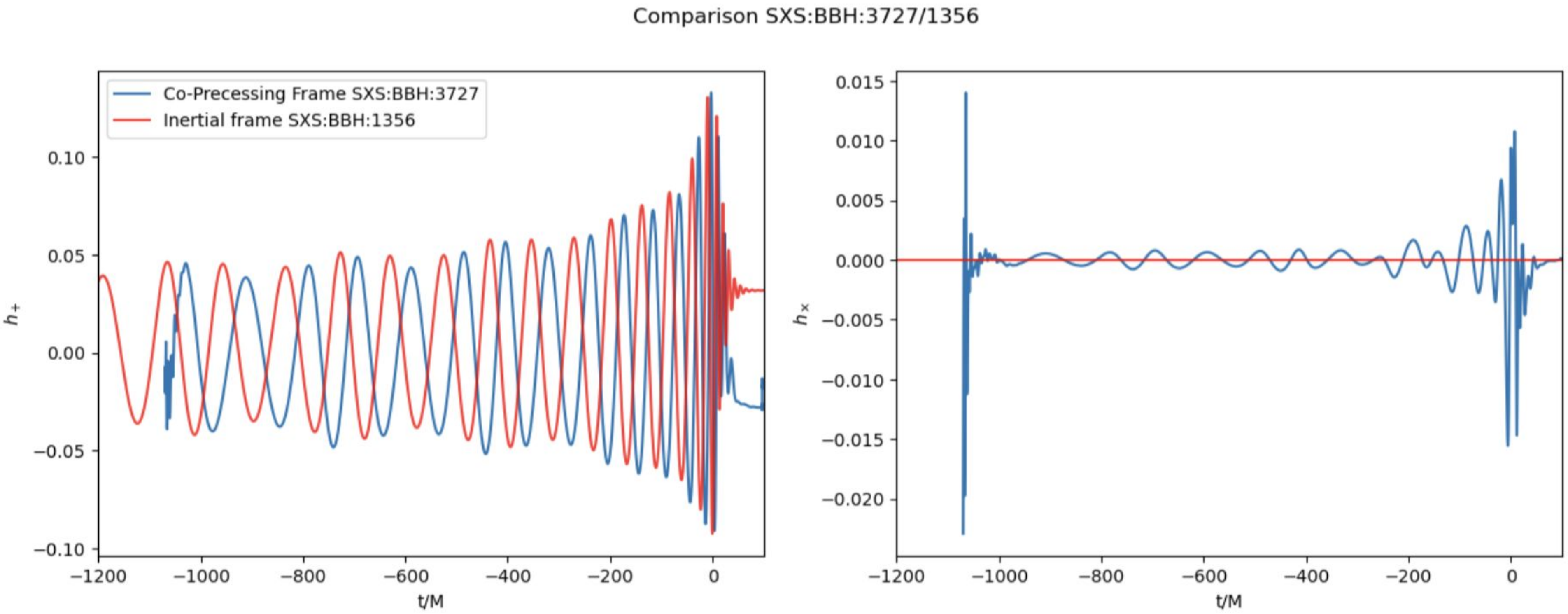
“Bad” Match Matching v1



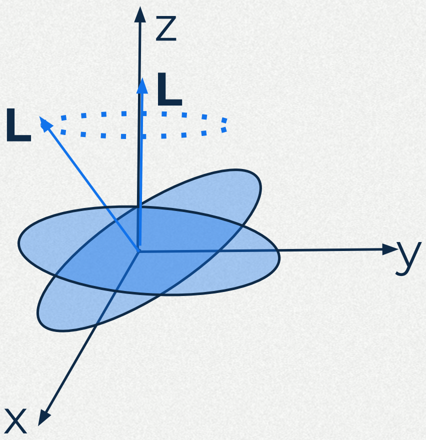
“Bad” Match Matching v2



“Bad” Match Matching v2



Finding the Co-Precessing Frame



L: orbital angular momentum vector

- Code package "scri" developed by Michael Boyle
- Uses method of O'Shaughnessy et al. (2011) to find *a* co-precessing frame
  - Finding dominant principal axis of matrix defined with orbital angular momentum operator
- Adds minimal rotation condition (specifies frame uniquely)
  - Algebraically minimizes instantaneous rotation vector
- Can refer to this transformation as "untwisting"

Minimal Rotation Condition

ertial frame, the time derivative of any vector stationary in the rotating frame is given by

$$\dot{\vec{v}} = \vec{\varpi} \times \vec{v}\ . \tag{1}$$

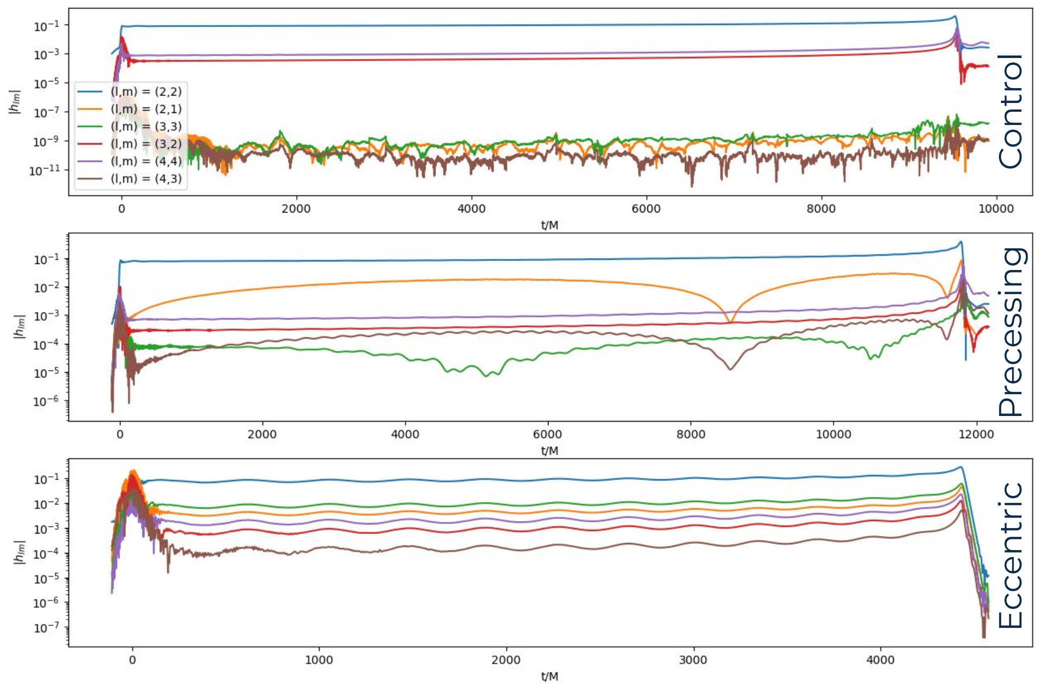
If we denote the radiation axis by a unit vector $\vec{a}$, we see that $\dot{\vec{a}} = \vec{\varpi} \times \vec{a}$. Taking the cross product of both sides of this equation with $\vec{a}$ and using the standard triple-product formula, we have

$$\vec{a} \times \dot{\vec{a}} = (\vec{a} \cdot \vec{a})\,\vec{\varpi} - (\vec{a} \cdot \vec{\varpi})\,\vec{a}\ . \tag{2}$$

Using the fact that $\vec{a}$ is unit, we can rearrange this as

$$\vec{\varpi} = \vec{a} \times \dot{\vec{a}} + (\vec{a} \cdot \vec{\varpi})\,\vec{a}\ . \tag{3}$$

Mode Hierarchies

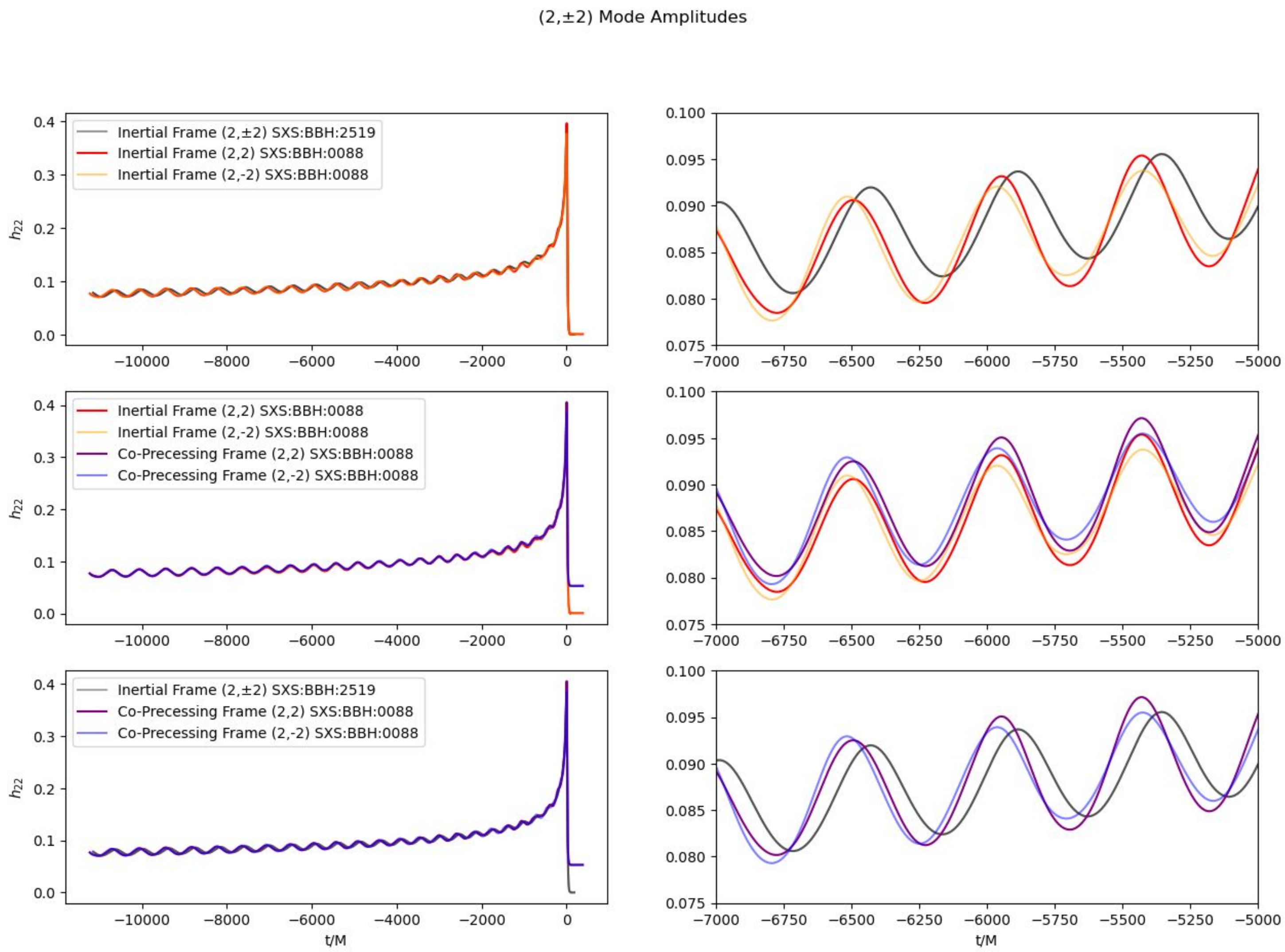


Parameter	Effect of parameter on mode hierarchy
Control(No spin, QC, q = 1)	(2,2) mode is strongest, (2,1) and (3,3) modes are zero.
Precession	Adds power to (2,1) mode.
Eccentricity	Adds power to (3,3) and (2,1) modes.
Mass Ratio	Adds power to (3,3) mode.

Mode Asymmetry

- In a non-precessing system, formula to the right holds
    - (l,m) and (l,-m) have same amplitudes
- Precession breaks that symmetry
    - orbital plane precesses during each period
    - breaks symmetry across plane on orbital timescale
- Mode asymmetry cannot be erased by any change of reference frame

$$h^{\ell,m} = (-1)^{\ell}\bar{h}^{\ell,-m}$$



(2,±2) Comparison

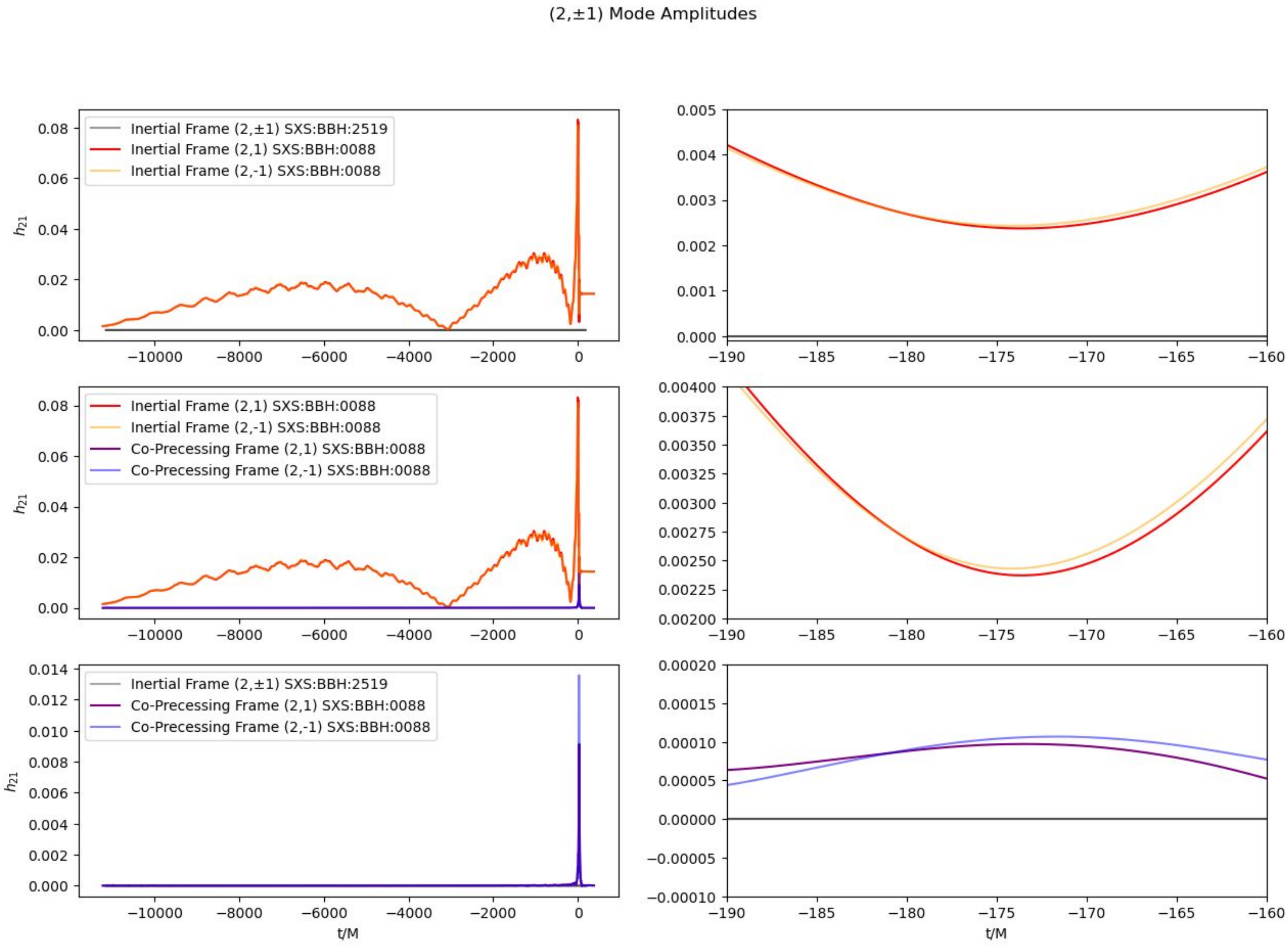
Darker shade = +m

Orange: 0088 inert

Purple: 0088 copr

Grey: Non-prec match

- copr frame moves envelope up to match non-prec counterpart
- mode asymmetries and phase remain
- orbital timescale effects unchanged by transformation?



(2,±1) Comparison

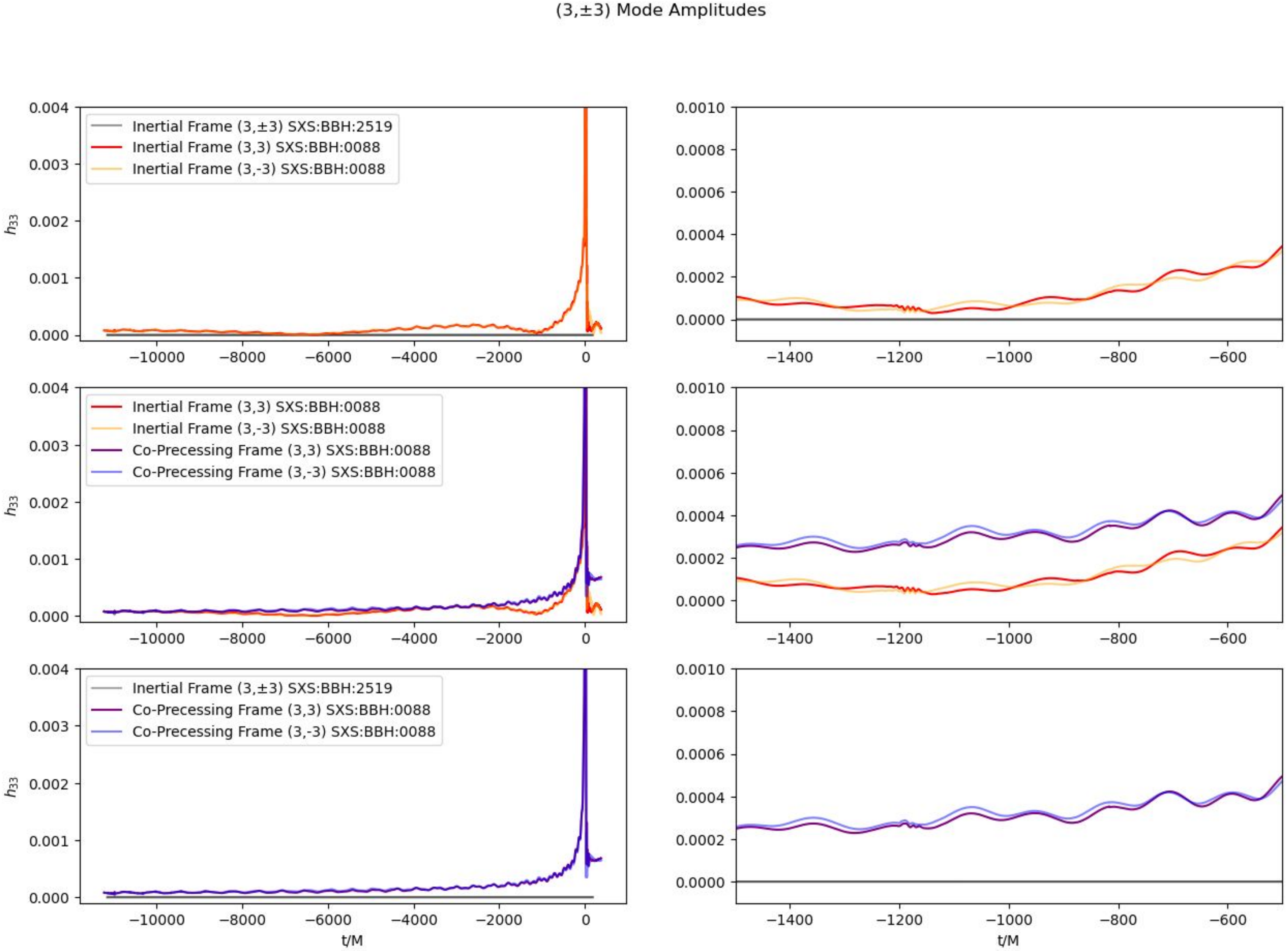
Darker shade = +m

Orange: 0088 inert

Purple: 0088 copr

Grey: Non-prec match

- copr frame brings amplitude much closer to 0
  - expected for n-p system
- flips concavity but not which mode is higher amplitude at a given time



(3,±3) Comparison

Darker shade = +m

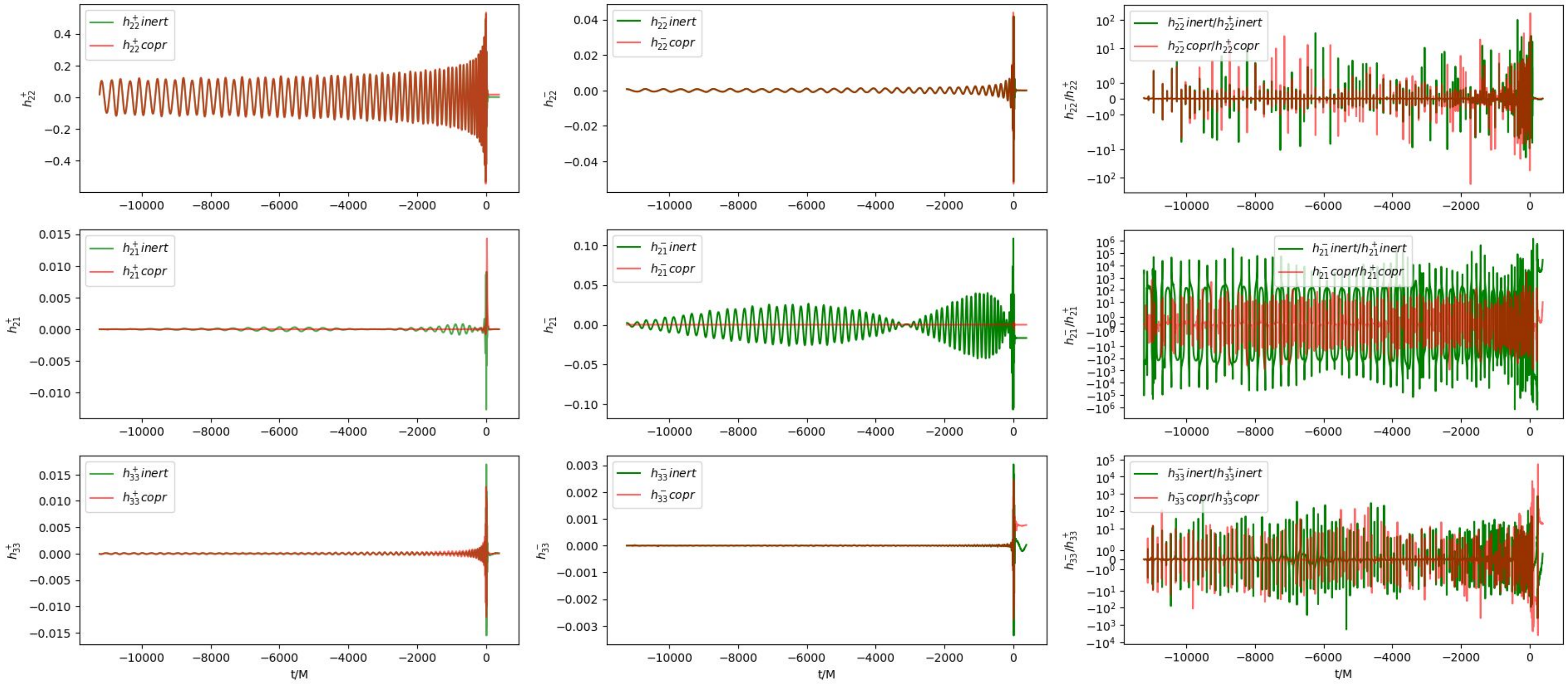
Orange: 0088 inert

Purple: 0088 copr

Grey: Non-prec match

- (3,3) amplitude *increases* in copr frame
- (2,1) mode isn't taking away power?
- again, ±m relationship stays through transformation

$$h^{\pm}_{\ell m} = \frac{h^{\text{coorb}}_{\ell,m} \pm h^{\text{coorb}\,*}_{\ell,-m}}{2}$$



Does copr frame transformation reduce mode asymmetry in (2,±1) modes?